Late time dynamics of scalar perturbations outside black holes.
I. A shell toy-model

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We present a new analytic approach for the study of late time evolution of linear test-fields, propagating on the exterior of black holes. This method provides a calculation scheme applicable to Kerr black holes (for which case no analytic calculation of the late time tails has been presented so far). In this paper we develop the new technique and apply it to the case of massless scalar waves evolving on the background geometry of a static spherically-symmetric thin shell with a Schwarzschild exterior. The late time behavior of the scalar field at null infinity is calculated, and is explicitly related to the form of (quite arbitrary) initial data. This reproduces the well-known late time power-law decaying tails. In an accompanying paper we apply our approach to the complete Schwarzschild black hole geometry, where we obtain the familiar inverse-power late time tails at null infinity, as well as at time-like infinity and along the event horizon. A detailed calculation of the late time power-law tails in the Kerr geometry, based on the same approach, will be presented in a forthcoming paper.

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I. INTRODUCTION

The gravitational field outside black holes created during a generic gravitational collapse relaxes to the stationary Kerr-Newman field. This result, referred to as the "*no-hair theorem*" (see, for example, [1]) implies that outgoing radiation should carry away all the initial characteristics of the collapsing object, except for its mass, charge, and angular momentum. These three quantities, characterizing the Kerr-Newman field, are conserved by conservation laws; any other quantity will vanish by the time the gravitation field settles down on its stationary state.

The underlying mechanism for this relaxation process was first demonstrated by Price [2] for the case of a nearly spherical collapse. Price analyzed the dynamics of massless integer-spin test fields, evolving on the background of a Schwarzschild black hole (SBH), and showed that when viewed from a fixed location outside the black hole, the waves die off at late time with an inverse power-law tail (in the Schwarzschild time t), whose power index depends only on the multipole number l of the mode under consideration. Later [3], it was demonstrated by analytic and numerical methods that the nearly spherical collapse exhibit late time decay tails also at future null infinity and along the event horizon. The formation of these tails was explained as due to back-scattering of the outgoing radiation off spacetime curvature at very large distances.

The existence of late time tails was demonstrated for linear perturbations of both SBH [2,3] and Reissner–Nordström [3] exteriors. Remarkably, numerical analysis of the fully non-linear dynamics of fields yields the same decay rates at late time, as for the minimally-coupled (linear) fields [4,5]. This, of course, encourages the application of the perturbative approach, even though the problem in discussion is non-linear in its nature.

Ching *et al.* [6] explored a wide class of asymptotically-flat spherically symmetric spacetimes, represented by effective curvature potentials of the form $(\ln r)^\beta / r^\alpha$ (where $\beta = 0, 1$ and $\alpha > 2$ are parameters). This class includes the SBH and Reissner-Nordström geometries. They found that, generically, the wave behavior at late time is not characterized by a strict power-law tail, but rather it has the form $(\ln t)^\beta \times$(inverse-power in t). In that respect, the Schwarzschild and Reissner-Nordström geometries represent a special subgroup of spacetimes. For the monopole moment of the scalar radiation, it was shown by Gómez *et al.* [7] that the form of the tails (whether "logarithmic" or not) depends on whether or not the Newman–Penrose constant for the field vanishes.

Recently [8], Brady *et al.* studied scalar waves dynamics in the non-asymptotically-flat exteriors of Schwarzschild–de Sitter and Reissner–Nordström–de Sitter black holes. Contrary to the asymptotically-flat geometries, no power-law tails were detected in these cases. Instead, the waves were found to decay exponentially at late time.

The aforementioned analysis by Ching *et al.* follows a technical scheme, first introduced by Leaver [9], in which the linear waves are first Fourier-decomposed, then evaluated in the complex frequency plane (see also [10]). In this technique, the late-time tails are explained in terms of a branch cut in the Green's function in the frequency domain. The fact that the branch cut is due to the form of the potential at asymptotically large radius (see [6] for details) implies, again, that the tails originate from scattering off the curvature potential at large distances. This observation, in turn, suggests that the development of tails is independent of the existence of an event hori-

zon. Thus the tail phenomenon appears to be of a more universal nature: it may characterize the realistic stellar dynamics. This, indeed, was suggested by Gundlach *et al.* [3,4] and Ching *et al.* [6]. In [4], for example, the purely spherical collapse of a self-gravitating minimally coupled scalar field was studied. It was demonstrated numerically that in this case late time tails are formed even when the collapse fails to create a black hole. We shall further discuss this issue in the present paper.

Historically, the study of wave dynamics outside black holes was motivated by the will to construct a detailed description of the relaxation process leading to the stationary "no hair" state (For example, there was a need to verify that the event horizon is indeed stable under generic perturbations). Many recent studies, however, are oriented by the prospects of directly observing gravitational radiation from astrophysical systems (e.g. with the LIGO observatory, see [11]). Although the study of late time behavior of fields outside black holes has probably no immediate observational implications (late time outflux from a realistic coalescing binary system, for example, would be orders of magnitudes weaker than the short pulse of radiation expected during the last few seconds of the merger), it still addresses several important questions. For example, this study has crucial relevance to the exploration of the internal structure of black holes, as it provides the input for the internal wave evolution problem. Particularly, it has been argued [12] that the form of the slowly decaying wave tail along the event horizon affects the strength of the mass-inflation singularity at the Cauchy horizon inside charged and rotating black holes.

The study of waves evolving in curved geometries is also interesting on its own right, from a theoretical point of view: such waves, massless just as well as massive, do not propagate along light cones solely; rather they also spread inside them. Regardless of the presence or absence of an event horizon, it is this feature of the evolution which is responsible for the phenomenon of late time decay tails in curved spacetimes.

In virtue of previous studies, we now have the following schematic picture regarding the dynamics of waves outside a nearly-spherical collapsing object: Consider a perturbation in the form of a compact pulse of radiation (gravitational or electromagnetic), somewhere outside the collapsing object, emitted at some time during the collapse. This pulse may represent radiation emerging from the surface of a collapsing object, as well as any other form of perturbation on the background geometry. We shall refer to it as the "initial pulse". A static observer outside the black hole will then indicate three successive stages of the wave evolution. First, the exact shape of the waves front depends on the detailed form of the initial pulse. This stage (lasting a period of time comparable to the duration of the initial pulse) is followed by a "quasinormal ringing" (QNR) relaxation stage, during which the waves undergo exponentially-decaying oscillations with (complex) frequencies completely determined

by the mass and electric charge of the central object [13]. Finally, as the QNR dies off exponentially in time, it leaves behind an inverse power-law decaying tail of radiation. During the last two stages, the details of the initial pulse affect the shape of the waves only through a global amplitude factor, hence the evolution during these stages is purely characteristic of the background geometry.

Now, a realistic black hole formed by a generic gravitational collapse is expected to spin, as do astrophysical stars (On the contrary, models of black holes with electric charge and/or cosmological constant are mainly hypothetical). Therefore it is natural to ask how the tail phenomenon is affected by the presence of angular momentum in the background geometry. One may suspect that the late time behavior of waves, evolving outside a rotating Kerr black hole, would be qualitatively of the same nature as in the spherically-symmetric spacetimes. This is because the Kerr geometry asymptotically approaches that of a SBH at very large radius, and presumably it is this far region of spacetime whose structure determines the form of the late time radiation.

Yet, in spite of the obvious interest, no analytic scheme has been proposed thus far to describe wave dynamics outside rotating black holes. The basic obstacle is, of course, the fact that the axially-symmetric Kerr black hole possesses three non-trivial dimensions, instead of only two in the spherical cases. This makes both analytic and numerical investigation significantly more complicated. Recently, there was an initial progress, with the introduction of a full $(2 + 1$ dimensions) numerical analysis by Krivan *et al.* [14,15], though much effort is still required is this direction.

The new calculation approach

We have developed an analytic calculation scheme enabling one to analyze late time wave dynamics outside Kerr black holes. This analysis shall be presented in [16] (see also [17]). The major goal of this paper, together with the accompanying paper (to be referred to as **Paper II**), is to demonstrate the applicability of our technique in a simpler model, namely the evolution of scalar waves in the SBH exterior. This will serve several purposes: First, we shall be able to test our scheme against the firm results already obtained for this case by previous studies. Secondly, many parts of the formalism to be developed shall be later directly employed in the analysis in Kerr. Finally, our analysis in Schwarzschild proves to be valuable on its own right, providing, in some respects, a more complete picture of the late-time wave behavior than already available.

Basically, our analysis is composed of two major steps. In the first and more crucial one, a characteristic initial value evolution problem for the scalar waves is treated analytically, resulting in the construction of a late-time solution for the wave *at null infinity*. This calculation in-

volves the introduction of a special perturbative decomposition of the waves, followed by the application of the standard (time domain) Green's function technique. In the second step we then use a simple late time expansion of the wave near time-like infinity, in order to obtain the late time behavior of the wave at *any* constant radius, including along the event horizon. This second step is made possible only after the waves' form at null-infinity is derived by the first step of the analysis.

To introduce the analysis at null infinity (i. e. the first step mentioned above) in a clear and more instructive way, we incorporate in this paper a simple toy-model, in which the scalar waves are taken to evolve in the gravitational field induced by a spherically-symmetric thin shell of matter. This configuration possesses the same spacetime structure at large distances (outside the shell) as in a complete SBH manifold of the same mass, while its small-r structure is much simpler. That would reduce the amount of technical details to deal with when developing our calculation scheme, and may also enable one to push the analytic calculation to a further extent, while leaving the essential features of the analysis unaffected. Indeed, we later show in **paper II**, considering the complete SBH manifold, that the asymptotic late time behavior calculated at null-infinity is unaffected by the structure of spacetime at small radii, thus it is essentially the same in the complete SBH manifold as it is in the shell toy-model.

Arrangement of this paper

In this paper we study the thin-shell toy model. To that end we introduce our calculation method, which we call "*the iterative scheme*", and apply it to calculate the behavior of the wave at null-infinity in this model.

The paper is arranged as follows: In sec. II we formulate the problem of wave evolution in the shell model as a characteristic initial-value problem. The iterative scheme is presented in sec. III. It is applied in sections IV through VIII, revealing the power-law pattern of the late time wave decay at null infinity. The amplitude of the wave at late time is explicitly calculated, expressed in terms of a (quite arbitrary) initial data function.

II. INITIAL-VALUE FORMULATION OF THE WAVE EVOLUTION PROBLEM

A. The shell model

We consider a static spherically-symmetric thin shell of matter, of a mass M and some radius $R > 2M$. The parameter R should be regarded as being of order $\sim 2M$ (say, $R = 3M$). For this configuration, the exterior vacuum region $r > R$ is a part of the Schwarzschild spacetime, described by the line element

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (1)$$

where t, r, θ and φ are the standard Schwarzschild coordinates and $f(r) \equiv (1 - 2M/r)$. The region inside the shell, $r < R$, is flat:

$$ds^2 = -dT^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (2)$$

Expressed in these coordinates, the geometry suffers a discontinuity at $r = R$ (the metric functions g_{rr} and g_{tt} jump through this surface. Also the time coordinates do not agree on both sides of the shell).

To allow a continuous representation of spacetime geometry, we define the coordinates

$$r_* = \begin{cases} f(R)^{-1/2}\, r & , r < R \\ r - R + Rf(R)^{-1/2} + 2M \ln\left(\frac{r-2M}{R-2M}\right) & , r > R \end{cases}$$

$$(3)$$

and

$$t_* = \begin{cases} f(R)^{-1/2}\, T & , r < R \\ t & , r > R. \end{cases} \quad (4)$$

In terms of the coordinates t_*, r_*, θ and ϕ the geometry is described as a single continuous manifold. (These coordinates form an example of a 'natural' set of coordinates through the 'surface layer' $r = R$, as discussed by Israel in [18]. The jump in the derivatives of the metric functions through the shell is then related to the surface energy tensor of this layer.) Note that outside the shell, the radial coordinate r_* is the usual Schwarzschild's 'tortoise' coordinate, satisfying $dr_*/dr = f^{-1}(r)$.

B. The initial-value problem

We consider the evolution of initial data, representing a generic pulse of massless scalar radiation, on the fixed background of the shell described above. The scalar field is assumed to satisfy the (minimally-coupled) Klein–Gordon equation[*]

$$\Phi_{;\mu}^{\;;\mu} = 0, \quad (5)$$

where Φ represents the scalar wave. The structure of spacetime affects the evolution of the scalar wave through the covariant derivatives, denoted in Eq. (5) by semicolons.

Decomposing the field in spherical harmonics,

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[*]We comment, however, that other wave equations are also possible. For example, to assure conformal invariance, one should rather use the equation $\Box\Phi + \frac{1}{6}R\Phi = 0$, where R is the Ricci scalar. This equation reduces to Eq. (5) when considering waves in vacuum.

$$\Phi(t,r,\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \phi^l(t,r) Y_{lm}(\theta,\varphi), \qquad (6)$$

and substituting in the wave equation (5), we obtain an independent equation for each of the components $\phi^l(t,r)$. (We use the superscript l to denote the multipole number of the mode under consideration.)

A convenient form for the wave equation may be obtained in terms of a new wave function $\Psi^l(t,r) \equiv r\phi^l(t,r)$. To that end we introduce the double-null (Eddington–Finkelstein) coordinates $v \equiv t + r_*$ and $u \equiv t - r_*$. The equation governing the evolution of the l-mode of the scalar wave then reads

$$\Psi^l_{,uv} + V^l(r)\Psi^l = 0, \qquad (7)$$

in which the function

$$V^l(r) \equiv \begin{cases} \frac{1}{4}l(l+1)r_*^{-2} & , r < R \\ \frac{1}{4}\left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right] & , r > R \end{cases} \qquad (8)$$

serves as an effective potential for the scalar field. The form of the potential function completely determines the effect of spacetime curvature on the wave evolution [terms proportional to M], as well as the centrifugal effect on the non-spherical ($l > 0$) modes of the wave [terms proportional to $l(l+1)$].

To set-up the initial value problem for the scalar radiation, one should supply initial data for each of its modes. Since in spherical symmetry each mode evolves separately, it would be sufficient to analyze the evolution of a single l-mode of the wave from a given initial data. We shall specify this initial data on two characteristic (null) surfaces, as illustrated in figure 1[†]. Specifically, we shall consider initial data in the form of some compact outgoing pulse, specified on the ingoing null surface $v = 0$:[‡]

$$\begin{cases} \Psi^l(u = u_0) = 0 \\ \Psi^l(v = 0) = \Gamma^l(u) \end{cases}, \qquad (9)$$

where $\Gamma^l(u)$ is some function of a compact support between retarded times $u = u_0$ and $u = u_1$ (for brevity we henceforth usually suppress the l-dependence of the functions under consideration). We shall assume that $|u_1| > 2R$, namely that the support of the initial function $\Gamma(u)$ is completely outside the shell.

The choice of compact ('localized') initial data proves to be convenient for the purpose of calculation. It also

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[†]For a characteristic initial value problem to be well posed, it suffices to specify only the value of the field on the initial characteristic surfaces (and not its derivatives).

[‡]The choice $v = 0$ does not limit the generality of the initial setup, due to the time translation invariance of the background geometry.

becomes useful when we later try to characterize the way in which the late time behavior of the wave depends on the location of the initial pulse. For that reason, it is instructive to consider the case $|u_0| \gg u_1 - u_0$, for which u_0 becomes a single parameter describing the location of the initial pulse.
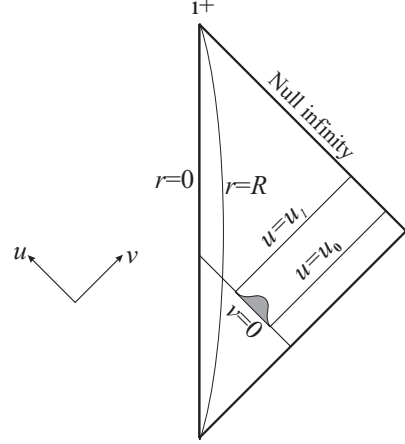

FIG. 1. The set-up of initial data. Shown is the conformal diagram representing spacetime geometry in the shell model. The spacetime region outside the shell ($r > R$) is a part of the Schwarzschild manifold, while the interior of the shell ($r < R$) is flat. $u = u_0$ and $v = 0$ are two initial characteristic (null) surfaces. An initially compact outgoing pulse is specified on $v = 0$.

To complete the specification of the initial value problem, a boundary condition should be set at the origin of coordinates. From a physical point of view, a wave starting out of regular initial data and traveling outside the shell, should not develop any singular behavior. Also, the field should remain regular (in fact, continuous) at the shell itself, since the jump in the potential function through this surface is finite in magnitude . Then, no singularity is expected to occur during the free evolution of the scalar field in the Minkowski interior of the shell. Enforcing regularity on ϕ then automatically implies that the revised wave function $\Psi = r\phi$ should vanish at the origin $r_* = r = 0$. To rule out unphysical divergent solutions, we hence impose the supplementary boundary condition

$$\Psi(r_* = 0) = 0. \qquad (10)$$

In appendix A we show that the evolution equation (7), together with the initial conditions (9) and the boundary condition (10), establish a well defined characteristic initial value problem for the scalar field, with a unique solution anywhere inside the light cone of the initial data.

III. THE ITERATIVE EXPANSION

Our goal is to calculate the late time behavior of Ψ at null-infinity (namely, for $v \to \infty$, $u \gg M$). In what

follows we introduce an analytic scheme to construct this solution. This technique is based on the *iterative expansion*, to be defined now.

First, define

$$V_0(r) \equiv \frac{l(l+1)}{4r_*^2} \qquad (11)$$

and

$$\delta V(r) \equiv V(r) - V_0(r), \qquad (12)$$

with the potential $V(r)$ given by Eq. (8).

Then, consider a decomposition of the wave function,

$$\Psi = \Psi_0 + \Psi_1 + \Psi_2 + \cdots, \qquad (13)$$

such that the components Ψ_N obey the recursion formula

$$\Psi_{N,uv} + V_0\Psi_N = \begin{cases} 0 & , N = 0 \\ -(\delta V)\Psi_{N-1} & , N > 0 \end{cases} \qquad (14)$$

and satisfy the initial conditions

$$\begin{aligned} \Psi_N(u = u_0) &= \quad 0 \quad (\forall N \geq 0) \\ \Psi_N(v = 0) &= \begin{cases} \Gamma(u) & , N = 0 \\ 0 & , N > 0 \end{cases} \end{aligned} \qquad (15)$$

and the boundary conditions

$$\Psi_N(r_* = 0) = 0 \quad (\forall N \geq 0). \qquad (16)$$

Formally summing Eqs. (14,15,16) over N, we recover Eqs. (7,9,10) for the complete wave Ψ. This suggests that if the sum (13) converges, it should yield the correct function Ψ.

Equations (14,15,16) constitute an infinite hierarchy of initial value problems for the functions Ψ_N. Each of the evolution equations consists of a simple homogeneous part, which is the free evolution equation in Minkowski's spacetime. In addition, each of the functions Ψ_N (excluding Ψ_0) has a source term proportional to the previous function in the series. Therefore, in principle, if the solution for Ψ_0 is found, and the appropriate (time domain) Green's function is constructed, then we should be able to solve for the functions Ψ_N one by one, using the standard Green's function method.

Analytic arguments, as well as numerical simulations (both to be presented later) strongly suggest that at null-infinity the series (13) *does* converge to the correct function Ψ. Moreover, in the case $|u_0| \gg M$, which will concern us here, we find (both numerically and analytically) that the late-time behavior of Ψ at null-infinity is well approximated by Ψ_1 (the corrections coming from $N \geq 2$ are smaller by a factor proportional to M/u_0). Thus, in our scheme, understanding the behavior Ψ_1 alone will suffice to determine the essential features of the late-time dynamics at null-infinity.

In the sequel we derive an exact analytic expression for Ψ_0 and for the Green's function , and use these results to calculate the late time behavior of Ψ_1 at null-infinity. We then discuss the contributions coming from higher orders of the iterative expansion.

IV. DERIVATION OF Ψ_0

By definition, Ψ_0 admits the homogeneous wave equation

$$\Psi_{0,uv} + V_0(r_*)\Psi_0 = 0, \qquad (17)$$

where V_0 is the purely centrifugal potential defined in Eq. (11). This equation is supplemented by the initial conditions

$$\Psi_0 = \begin{cases} 0 & , u = u_0 \\ \Gamma(u) & , v = 0 \end{cases}, \qquad (18)$$

and by the boundary condition specified in Eq. (16). (recall that Γ is a function of compact support, representing the initial pulse of scalar radiation.)

In fact, Ψ_0 is nothing but the solution of the analogous wave evolution problem in *Minkowski* spacetime[§]. To see that, notice that inside the shell Eq. (17) is invariant under the transformation from the r_*, t_* coordinates to the usual flat-space coordinates r and t. Outside the shell, however, only the *functional* form of $\Psi_0(u, v)$ is the same as in Minkowski, while the dependence of the characteristic coordinates u, v on the flat space coordinates r, t differs.

In appendix A we show that a solution Ψ_0 to Eq. (17), subject to both the initial conditions (18) and the boundary condition at $r = 0$, is *unique*. This solution reads

$$\Psi_0(u_0 \leq u \leq 0) = \sum_{n=0}^{l} A_n^l \frac{g_0^{(n)}(u)}{(v-u)^{l-n}} \qquad (19a)$$

$$\Psi_0(u \geq 0) \equiv 0, \qquad (19b)$$

in which the coefficients A_n^l are given by

$$A_n^l = \frac{(2l-n)!}{n!(l-n)!}. \qquad (20)$$

The function $g_0(u)$ (with its parenthetical superindices indicating the number of times this function is differentiated) is, by Eq. (18), the solution to the inhomogeneous ordinary equation of order l,

$$\sum_{n=0}^{l} A_n^l \frac{g_0^{(n)}(u)}{(-u)^{l-n}} = \Gamma(u), \qquad (21)$$

subject to the initial conditions $g_0^{(n)}(u_0) = 0$ for all $0 \leq n \leq l-1$. (Due to the compactness of $\Gamma(u)$, we then automatically have also $g^{(l)}(u_0) = 0$. Thus Ψ_0 vanishes along the ray $u = u_0$, as necessary.)

[§] In this respect, the iterative decomposition is actually "an expansion of Schwarzschild about Minkowski".

The solution for g_0 is given by

$$g_0(u \leq 0) = \frac{1}{(l-1)!} \int_{u_0}^{u} \left(\frac{u}{u'}\right)^{l+1} (u-u')^{l-1} \Gamma(u') du'$$

$$(22)$$

for $l \geq 1$, and simply $g_0(u) = \Gamma(u)$ for $l = 0^{**}$. It is easy to confirm this result by a direct substitution, noticing that a solution to the homogeneous equation corresponding to Eq. (21) has the general form

$$g_0(u_1 \leq u \leq 0) = \begin{cases} 0 & , l = 0 \\ \sum_{k=1}^{l} \gamma_k u^{l+k} & , l > 0 \end{cases}, \qquad (23)$$

where γ_k are constant coefficients. Note that this is also the form of g_0 at $u_1 \leq u \leq 0$ (for which retarded times Eq. (21) becomes homogeneous), with the coefficients γ_k being certain functionals of the initial data function Γ, that can be easily constructed by comparing Eqs. (22) and (23).

We find that during retarded times $u_0 \leq u \leq u_1$, at which the initial pulse is "turn on" on the initial ingoing null surface, the behavior of $\Psi_0(u)$ depends on the detailed structure of the pulse. At later (yet negative) retarded times, $u_1 \leq u \leq 0$, the function $g_0(u)$ takes the simple form $u^{l+1} \times$ polynomial of order $l-1$ in u. Eq. (19a) then implies that the wave Ψ_0 itself dies off (as $\sim u$, generically) towards retarded time $u = 0$. Exceptional is the $l = 0$ mode of the wave, which vanishes right after the initial pulse 'ceases' on the initial ray (that is, it vanishes identically at $u \geq u_1$).

In the later zone of spacetime, $u > 0$, the only solution for Ψ_0 satisfying both $\Psi_0(u = 0) = 0^{\dagger\dagger}$ and $\Psi_0(r_* = 0) = 0$, is the trivial null solution (19b) [see figure 2]. The uniqueness of this solution is guaranteed in virtue of the discussion in Appendix A.

We can now unify both Eqs. (19) by defining

$$g_0(u > 0) \equiv 0. \qquad (24)$$

It is also convenient to define $g_0(u < 0) \equiv 0$, which makes $g_0(u)$ a function of compact support between $u = u_0$ and $u = 0$. Then, Eq. (19a) suffices to describes the wave Ψ_0 anywhere.

**In the scalar model, the monopole mode is radiative as any other mode.

†† Here we assert that the wave Ψ_0 is continuous through the ray $u = 0$. For, as can be easily verified, a discontinuity along this ray must result is a violation of either the field equation or the boundary condition. The continuity of the wave is also reasonable from the physical point of view.
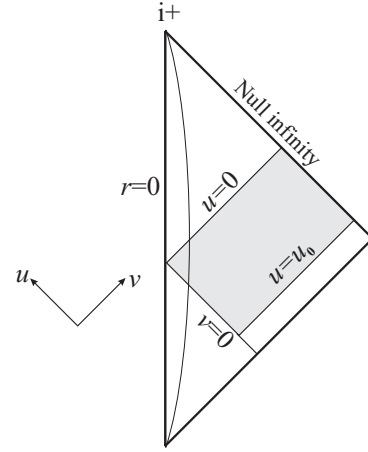

FIG. 2. The evolution of Ψ_0 (representing the scalar wave in Minkowski spacetime). For a compact initial pulse 'turned on' during retarded times $u_0 < u < u_1 < 0$, the support of $\Psi_0(u)$ is confined to $u_0 < u < 0$ (or $u_0 < u < u_1$ for $l = 0$). The region in which Ψ_0 has non-zero amplitude (for $l \geq 1$) is represented by the shadowed area in this conformal diagram.

We conclude that for any mode of the scalar radiation, the evolution of Ψ_0 is "cut off" not later then at $u = 0$. This somewhat surprising feature of the scattering off the purely centrifugal potential originates, in the 1+1 representation, from a destructive interference between ingoing and outgoing wave fronts at the origin $r_* = 0$ (see appendix A for details). This may be more easily understood in the corresponding 3+1 picture, where compactness of the initial pulse directly leads to compactness of the wave fronts. The compact (in terms of retarded time) region in which Ψ_0 survives is represented by the dark-colored area in the diagram of figure 2.

Finally, we emphasize one further feature of Ψ_0. By Eq. (19a), we have *at null-infinity*

$$\Psi_0^{\infty}(u) = g_0^{(l)}(u) \qquad (25)$$

in which $\Psi_0^{\infty}(u)$ stands for $\Psi_0(v \to \infty)$. Now, the fact that $g_0(u)$ is compact with respect to retarded time, means that any of its derivatives is compact as well. We thus find that the l integrals $(\Psi_0^{\infty})^{(-1)}$, ..., $(\Psi_0^{\infty})^{(-l)}$, carried out over all values of u at null infinity, *all vanish*:

$$(\Psi_0^{\infty})^{(-n)} = g_0^{(l-n)}\Big|_{-\infty}^{\infty} = 0 \quad (1 \leq n \leq l). \qquad (26)$$

This feature shall appear to have an important impact on the form of the inverse power-law behavior of the scalar wave at late time.

Obviously, since Ψ_0 vanishes identically at $u > 0$, it does not contribute to the late time radiation. Rather, it serves as a source to higher terms in our iterative expansion, as we show below.

V. CONSTRUCTION OF THE GREEN'S FUNCTION

In order to calculate the next terms in the iterative expansion (that is to obtain solutions to the hierarchy of inhomogeneous equations (14) for $N \geq 1$), we shall use the standard Green's function approach. To that end, we first need to obtain the Green's function corresponding to the operator $\partial_v \partial_u + V_0^l$. This is the purpose of this section.

The (retarded) Green's function $G(u, v; u', v')$ shall be defined as a solution of the equation

$$G_{,uv} + V_0^l G = \delta(u - u')\delta(v - v'), \qquad (27)$$

subject to the causality condition $G = 0$ outside the future light cone of the delta source point at (u', v'). In the sequel it will become evident that this condition, together with an appropriate boundary condition, specify a unique solution for G.

To construct this solution we consider separately the two distinct regions of spacetime indicated in figure 3, which are defined with respect to given a source point (representing a source 2-sphere in 3+1 dimensions) at a certain location (u', v'). These are (I) the region inside the future light cone of (u', v') not causally influenced by the origin (that is $v \geq v'$, $u' \leq u \leq v'$); and (II) the region inside the future light cone of (u', v') which *is* causally influenced by the origin (that is $u > v'$).

First consider the Green's function in region I. We impose causality by writing

$$G = \bar{G}\, \theta(u - u')\, \theta(v - v'), \qquad (28)$$

where $\bar{G}$ is a solution to the homogeneous equation (27) at $u > u'$ and $v > v'$, and θ is the standard step function. By analogy to Eq. (19a) we then have the solution

$$\bar{G}(u, v; u', v') = \sum_{n=0}^{l} A_n^l \frac{g_G^{(n)}(u)}{(v - u)^{l-n}}, \qquad (29)$$

in which the coefficients A_n^l are those given by Eq. (20), and where the function $g_G(u; u', v')$ is yet to be determined. (Alternatively, we could have expressed this solution in terms of a function $h_G(-v)$ instead of $g(u)$. Of course, the unique solution for G, to be derived below, would have been the same).

Inserting Eq. (28) into Eq. (27), we find that we must have

$$\bar{G}(u = u') = \bar{G}(v = v') = 1. \qquad (30)$$

With Eq. (29), the equality $\bar{G}(v = v') = 1$ takes the form of Eq. (21), in which we first make the replacements $g_0 \to g_G$, $u \to (u - v')$ and $\Gamma(u) \to 1$. Consequently, we find by Eq. (22) that (in region I)

$$g_G(u; u', v') = \frac{1}{(l-1)!} \int_{u'}^{u} \left(\frac{u - v'}{u'' - v'}\right)^{l+1} (u - u'')^{l-1} du''. \qquad (31)$$

To calculate the integral we make use of the formula

$$\int \frac{(x - x_1)^{n_1}}{(x - x_2)^{n_2}} dx =$$
$$-\sum_{j=0}^{n_1} \frac{n_1!(n_2 - j - 2)!}{(n_1 - j)!(n_2 - 1)!} \frac{(x - x_1)^{n_1 - j}}{(x - x_2)^{n_2 - j - 1}}, \qquad (32)$$

which is valid for any two natural numbers n_1 and n_2 satisfying $n_2 \geq n_1 + 2$. (we shall need this integral several more times in the sequel). With some additional algebraic manipulations we can then finally obtain

$$g_G(u; u', v') = \frac{1}{l!} \left[\frac{(v' - u)(u - u')}{(v' - u')}\right]^l. \qquad (33)$$

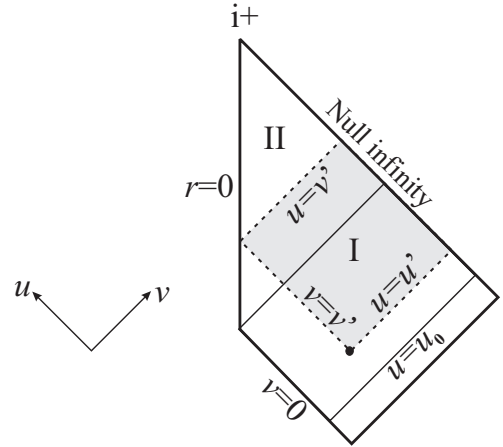It is easy to verify that with this result, we have also $G(u = u') = 1$, as necessary.

FIG. 3. The Green's function for a given source 2-sphere at (u', v'). G differs from zero only in region I (the shadowed area). It vanishes in region II.

Next, consider region II, where each of the functions Ψ_N is subject to the boundary condition $\Psi_N(r_* = 0) = 0$. We impose this condition by requiring that the Green's function connecting any source point to the origin should vanish:

$$G(r_* = 0; u', v') = 0. \qquad (34)$$

Note that this boundary condition fails to be valid at the single point $u = v = v'$, where its value, dictated by the evolution in region I, is found by Eqs. (29) and (31) to be unity. Nevertheless, the boundary condition for the various functions Ψ_N is still satisfied at all values of u, as will become clear later.

In appendix A we show that, with the above boundary condition imposed, the only solution to Eq. (27) in region II is the null solution, namely $G(u > v') \equiv 0$. We conclude that the Green's function is given by

$$G(u, v; u', v') = \sum_{n=0}^{l} A_n^l \frac{g_G^{(n)}(u; u', v')}{(v - u)^{l-n}}$$
$$\times \theta(v - v')\theta(u - u')\theta(v' - u), \qquad (35)$$

where the function g_G is the one specified in Eq. (33).[‡‡]

In figure 3 we have indicated the region of spacetime to which the Green's function $G(u, v; , u', v')$ carries the influence of a given source point located at (u', v'). In practice, however, we shall be interested in calculating the functions $\Psi_N(u, v)$ (for $N \geq 1$) at a certain evaluation point (u, v). This will involve integration over all point sources (u', v'). The region in which sources influence the behavior of the wave at a given evaluation point (u, v) is indicated in figure 4.
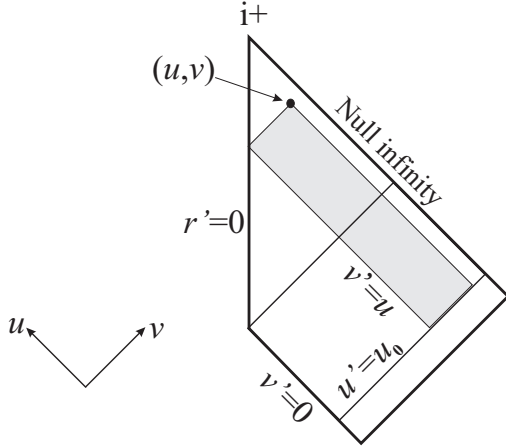
FIG. 4. The Green's function for a given evaluation point at (u, v). The wave Ψ_N at that point is influenced only by Ψ_{N-1} sources inside the shadowed area.

VI. CALCULATION OF Ψ_1 AT NULL INFINITY

We have seen that the first element of our iterative series, namely Ψ_0, do not contribute to the overall late time radiation. Rather, it serves as a source to higher terms in the series. The first contribution to the late time radiation comes from Ψ_1. In what follows we shall give a detailed calculation of Ψ_1. Specifically, we will obtain an analytic expression for Ψ_1 at null-infinity, evaluated to the leading order in M/u. The result will be highly significant, since this term will appear to be the dominant constituent of the overall wave Ψ at null infinity. We shall discuss this issue later, when we analyze the higher ($N \geq 2$) terms of the iterative expansion.

For Ψ_1 we have, by definition,

$$\Psi_{1,uv} + V_0 \Psi_1 = -(\delta V)\Psi_0, \qquad (36)$$

[‡‡] Note that the Green's function suffers a discontinuity along the outgoing ray $u = v'$. This discontinuity, which is of order unity, originates at the delta source point (u', v') and travels along the ingoing ray $v = v'$. When this discontinuity encounters the origin, it is reflected back along $u = v'$. Some subtleties related with the appearance of this discontinuity are discussed is appendix A.

with the initial conditions $\Psi_1(u = u_0) = \Psi_1(v = 0) = 0$, and the boundary condition $\Psi_1(r_* = 0) = 0$. The functions $V_0(r)$ and $\delta V(r)$ were given in Eqs. (11) and (12), respectively.

The solution can be formally written as

$$\Psi_1(u, v) = -\int_{u_0}^{u} du' \int_{u}^{v} dv' \, G(u, v; u', v') \delta V(u', v') \Psi_0(u', v')$$

(37)

[with $\Psi_0(v' < 0) \equiv 0$ understood], where G is the Green's function in Minkowski's spacetime, given in Eq. (35). This form manifestly admits the above initial conditions. Also, the boundary condition is clearly satisfied, as necessary. Notice that although Eq. (34) fails to hold at $u = v = v'$, still the wave Ψ_1 vanishes at that point [because the v' integration in Eq. (37) has no support in this occasion.]

The region of integration in Eq. (37), for a given evaluation point (u, v), is represented by the rectangle shown in figure 4. Sources outside this rectangle do not influence the behavior of Ψ_1 at (u, v). In addition, we have found previously that the support of Ψ_0 is confined to retarded times $u_0 < u < 0$. Hence, effective sources to Ψ_1 are located only at the region indicated in figure 5 by the intersection of the two shadowed areas. We thus observe that when evaluated at late retarded time u, Ψ_1 is influenced only by sources at large radii. As $u \to \infty$, it is only the asymptotically-far region of spacetime that affects the behavior of Ψ_1.
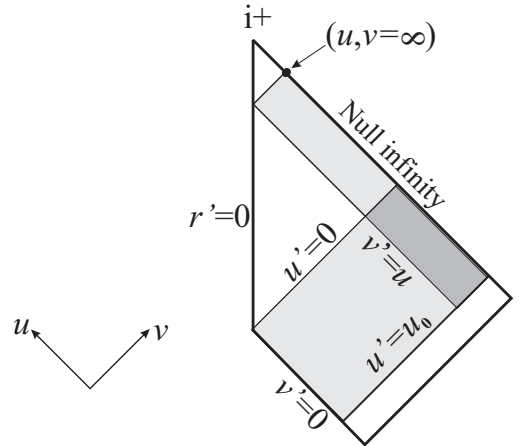

FIG. 5. Region where Ψ_0 sources affect Ψ_1 at null infinity (indicated by the intersection of the two dark colored areas). If Ψ_1 is evaluated at late retarded time $(u \gg M)$ then effective sources are confined to large distances $(r' \gg M)$.

We shall now evaluate Eq. (37), in order to yield an analytic expression for Ψ_1 at null infinity, that is in the limit $v \to \infty$. In this limit, the Green's function reduces to the form

$$G(u, v \to \infty; u', v') = \frac{\partial^l}{\partial u^l}[g_G(u; u', v')]. \qquad (38)$$

with only the $n = l$ term surviving in Eq. (35), and where g_G is given by Eq. (33).

We then notice that the l-derivative of g_G with respect to u can be taken out of the double integral in Eq. (37). The resulting surface terms all vanish, due to the factors $(u - u')^l$ and $(v' - u)^l$ appearing in the numerator of g_G [see Eq. (33)]. Therefore, in analogy to Eq. (25), we can now have

$$\Psi_1^\infty(u) = \frac{\partial^l}{\partial u^l}[g_1^\infty(u)], \qquad (39)$$

where we define

$$g_1(u,v) = -\int_{u_0}^u du' \int_u^v dv' \, g_G(u; u', v')\delta V(u', v')\Psi_0(u', v'), \qquad (40)$$

and where Ψ_1^∞ and g_1^∞ stands for the value of these function at null infinity $(v \to \infty)$. It would hence be sufficient to calculate g_1^∞ in order to immediately obtain Ψ_1^∞.

To allow explicit integration in Eq. (40), we must first express the function $\delta V(r)$ in terms of the null coordinates. This cannot be done explicitly outside the shell, since the function $r(r_*)$ is implicit at $r > R$. However, in terms of the M/r_* expansion we can write (Recalling that the parameter R is of order M)

$$\delta V(r_* > r_R) = \frac{a + b \ln 2\tilde{r}_*}{8r_*^3} + \emptyset\left(\frac{M^2(\ln \tilde{r}_*)^2}{r_*^4}\right), \quad (41)$$

where $\tilde{r}_* \equiv r_*/(R - 2M)$, a is some constant (depending on R and M), and

$$b = 8Ml(l+1). \qquad (42)$$

Throughout the rest of this paper, the symbol "$\sim$", when appearing over a character, shall always indicate the ratio of that quantity to $R - 2M$.

Note that inside the shell we have $\delta V \equiv 0$ by definition, and so the region $r'_* < r_*(R) = Rf(R)^{-1/2}$ is automatically excluded from the domain of integration in Eq. (40).

The various terms in the M/r_* expansion of δV contribute to Ψ_1 in an additive way [via Eq. (40)]. These contributions may be, in principle, calculated for each of the terms in separate. However, the calculation to follow suggests that to the leading order in M/u, it is only the leading order of δV in M/r_* which contributes to Ψ_1 at null infinity at late time . We shall therefore focus now on the contribution coming from this leading order.

With the explicit form of g_G and Ψ_0, and taking δV to the leading order in M/r_*, Eq. (40) reads at null infinity

$$g_1^\infty(u) = -\frac{1}{l!}\sum_{n=0}^l A_n^l \int_{u_0}^u du' g_0^{(n)}(u') \int_u^\infty dv' \frac{(v' - u)^l(u - u')^l}{(v' - u')^{2l-n+3}}$$
$$\times \, [a + b\ln(\tilde{v}' - \tilde{u}')]\,\theta[r'_* - r_*(R)], \qquad (43)$$

where the fact that $\delta V = 0$ inside the shell accounts for the step function appearing in the integrand.

To proceed, we discuss separately the cases $l = 0$ and $l \geq 1$. First, consider **the case $l \geq 1$**. For these modes we show that the 'r_*^{-3} potential' source (i.e. the one proportional to a) contributes nothing to g_1^∞ at $u \geq 0$. An analytic expression for the contribution coming from the 'logarithmic potential' source (the one proportional to b) shall then be derived.

To show that contribution from scattering off the r_*^{-3} potential vanishes (in the case $l \geq 1$), consider the term in (43) proportional to a. Integrating over v', with the help of Eq. (32), this term yields

$$a\sum_{n=0}^l B_n^l \int_{u_0}^u du'(u - u')^{n-2}g_0^{(n)}(u'), \qquad (44)$$

with a vanishing contribution from the upper boundary of the v'-integration, and where only the $j = l$ term has survived in Eq. (32). The coefficients B_n^l are given by

$$B_n^l = \frac{l - n + 1}{n!(2l - n + 1)(2l - n + 2)}. \qquad (45)$$

We now observe that in Eq. (44) all terms corresponding to $2 \leq n \leq l$ vanish independently for $u \geq 0$, since $g_0(u)$ is compact. This can be verified by integrating each of the terms in the sum by parts $n - 2$ successive times with respect to u', and then using Eq. (26) to find that all resulting surface terms vanish. Moreover, one also finds that the two remaining terms of Eq. (44), corresponding to $n = 0$ and $n = 1$, add to zero. This is easily shown by noticing that $B_0^l = B_1^l$ for any $l \geq 1$.

We conclude that all terms proportional to a in Eq. (43) vanish (for $u \geq 0$), and thus that scattering off the r_*^{-3} potential does not affect the late time behavior of the $l \geq 1$ modes at null-infinity. This seemingly odd feature becomes clear by the following simple argument. Consider the solution Ψ_0^ϵ to the wave equation (17), in which we take the potential to be $V_0^\epsilon = \frac{l(l+1)}{4(r_*+\epsilon)^2}$. In addition, we take Ψ_0^ϵ to satisfy the same initial conditions as Ψ_0, and the boundary condition $\Psi_0^\epsilon(r_* = \epsilon) = 0$. Clearly, we have $\Psi_0^\epsilon(\epsilon \to 0) = \Psi_0$. Differentiating the wave equation for Ψ_0^ϵ with respect to ϵ and taking the limit $\epsilon \to 0$, we obtain

$$\Psi'_{0,uv} + V_0\Psi'_0 = \frac{l(l+1)}{2r_*^3}\Psi_0, \qquad (46)$$

where $\Psi'_0 \equiv \lim_{\epsilon \to 0}\frac{\partial \Psi_0^\epsilon}{\partial \epsilon}$. Comparison of Eq. (46) with the wave equation (36) for Ψ_1, shows that Ψ'_0 should be proportional to Ψ_1 calculated with $\delta V \sim r_*^{-3}$. (Both functions, Ψ_1 and Ψ'_0, are subject to the same initial conditions: on the initial surfaces we have, by definition, $\Psi_1 = 0$ and also $\Psi'_0 = 0$, since the initial conditions chosen for Ψ_0^ϵ are ϵ-independent.) Now, it is not difficult to verify that Ψ'_0 vanishes identically at $u \geq 0$ (as does

$\Psi_0)^{\S\S}$. Therefore the contribution to Ψ_1 due to scattering off the r_*^{-3} potential must vanish identically at $u \geq 0$, as demonstrated above for Ψ_1^∞ by explicit calculation.

Next, we have to consider contributions to Ψ_1^∞ coming from scattering off the 'logarithmic' potential (i.e. contributions to the integral in Eq. (43) due to terms proportional to b). The details of this calculation are left to appendix B, where we obtain the simple result

$$g_1^\infty(u) = -2M \int_{u_0}^0 \frac{g_0(u')}{(u-u')^2} du'. \qquad (47)$$

The case $l = 0$: The monopole case is especially simple to handle, as in this case no logarithmic terms are involved in the calculation (the coefficient b in Eq. (41) vanishes), and the Green's function is simply unity. For $l = 0$ we have no summation over n in Eq. (43), and have $a = 4M$. This equation then reads

$$g_1^\infty(u) = -4M \int_{u_0}^0 du' \int_u^\infty dv' \frac{g_0(u')}{(v'-u')^3} \quad \text{(for } l=0\text{)}, \qquad (48)$$

which directly leads to Eq. (47). Therefore Eq. (47) is correct for any of the modes. (Note the interesting result that g_1^∞ depends on l only through the explicit form of g_0).

Thus far we were approximating the potential δV by its large r_* form, indicated in Eq. (41). In principle, to obtain the correct expression for g_1^∞, the complete potential δV should be considered. Each of the terms in the M/r_* expansion of the potential δV contributes additively as a source to $g_1(u)$. These terms are all proportional to $r_*^{-k_2}(\ln \tilde r_*)^{k_1}$, where k_1 and k_2 are natural numbers, satisfying $k_2 \geq 3$ and $0 \leq k_1 \leq k_2 - 2$. Above we have given a complete analysis concerning the leading contribution, namely the one corresponding to $k_2 = 3$. We can show, following a completely analogous analytic treatment, that the contribution from the above general term of the δV expansion (with $k_2 > 3$) is dominated at late retarded time by

$$g_1^\infty(u; k_1, k_2) \propto M^{k_2-2} \int_{u_0}^0 g_0(u') \frac{\ln^{k_1}(\tilde u - \tilde u')}{(u-u')^{k_2-1}} du'. \quad (49)$$

(Note that the $k_2 = 3$ case is special, in that in this case there is no logarithmic dependence involved in the resulting form of g_1^∞.) We thus conclude that omitting the contributions from terms other than the leading term in δV, has no effect on $g_1^\infty(u)$ to the leading order in M/u and in u_0/u.

---

§§ A straightforward calculation shows that Ψ_0^ϵ dies off at $u > 2\epsilon$ as $\sim \exp[-u/(2\epsilon)]$. This implies the vanishing of $\partial\Psi_0^\epsilon/\partial\epsilon$ at $u \geq 0$ in the limit $\epsilon \to 0$.

Hence, in virtue of Eq. (47) we obtain, to the leading order in M/u and in u_0/u,

$$g_1^\infty(u \gg u_0) = -2MI_0\, u^{-2}, \qquad (50)$$

in which we have defined

$$\int_{u_0}^0 g_0(u')du' = \int_{-\infty}^{+\infty} g_0(u')du' \equiv I_0. \qquad (51)$$

(The first equality is valid in view of the compactness of $g_0(u)$.)

By Eq. (39) we can now finally obtain (to the leading order in M/u and in u_0/u)

$$\Psi_1^\infty(u \gg u_0) = 2(-1)^{l+1}(l+1)!MI_0\, u^{-l-2}. \qquad (52)$$

Note that initial data information is manifested at null infinity at late time only through the single integral I_0. Also, note that in our approximation there is no reference whatsoever in the form of Ψ_1 to the radius R of the shell.

Both the power index of the radiation tail and the amplitude coefficient deduced above, are the same as those obtained by Gundlach, Price and Pullin (GPP)(Eqs. (11) and (18) in [3]) for the late time behavior of the 'complete' scalar wave at null-infinity, Ψ^∞, using a different approach. Yet, our result regards only one element of the 'complete' wave, namely Ψ_1^∞. To reveal the overall behavior of the wave we must analyze each of the terms in the iterative scheme, then sum up their contributions. Thus, in order for our result to coincide with that of GPP, Ψ_2^∞ and the higher terms in our expansion 'must' be negligible at null-infinity. In the following chapter we argue that this, however, is not the case in general. We will show that when considering generic initial date, one finds additional unnegligible contributions to Ψ^∞ at late time. These contributions, coming from $\Psi_2, \Psi_3, \ldots$, shall have the form $M^2 u_0^{-1} u^{-l-2}$, $M^3 u_0^{-2} u^{-l-2}$, etc. In this respect, the result of GPP is not strictly correct: the amplitude calculated in [3] should fail to represent the amplitude of Ψ^∞ for a generic initial pulse. However, as we indicate later, the result of GPP, as well as the form of Ψ_1^∞ calculated above, approximates the 'real' behavior of the scalar wave in the case where the initial compact pulse is confined to large radii, away from the highly curved region of spacetime. Mathematically speaking, it will be argued that the equality $\Psi_1 = \Psi$ holds at null-infinity to only the leading order in M/u_0.

It is interesting to mention that other previous attempts to calculate the late–time behavior of scalar waves in Schwarzschild [2,6,9,10], all yielded an expression proportional to the mass M, omitting possible contributions from higher powers of M. To our best knowledge, no analytic method has been proposed to enable calculation of the late time radiation resulting from initial date located at small radii, where curvature is large. In this respect, our iterative scheme is no exception. However, our method *does* provide a formal mean to quantitatively explore this aspect of the analysis, which was somewhat

overlooked in several previous works. We shall further refer to this issue later in this paper.

The expression we obtained above for Ψ_1^∞ [Eq. (52)] corresponds to the case of an initial pulse which is compact with respect to retarded time u. This pulse represents a wide family of physically reasonable initial data. Yet another realistic initial setup involves the presence of a static field outside some radius (say, the surface of a spherically collapsing object) up to some moment of time (say, the onset of collapse). The corresponding late time behavior of Ψ_1 at null infinity can be easily inferred from our previous results, as follows.

The static flat-spacetime solution to the Klein–Gordon equation (17), regular at infinity, reads $\Psi_0^{\text{stat}} = \mu r^{-l}$, where μ is a constant representing the strength of the initial static field. The corresponding g_0 function, namely the solution to Eq. (21) with $\Gamma(u) = \Psi_0^{\text{stat}}(v = 0)$, is $g_0^{\text{stat}} = 2^l [l!/(2l)!]\mu$. In order to calculate Ψ_1 in this case, we need only to substitute for g_0 in Eq. (47), while changing the range of integration to $\int_{-\infty}^{u_1} du'$, u_1 being some retarded time after the initial pulse ceases on the initial ingoing ray. Doing so, we find, to the leading order in M/u and in u_1/u,

$$g_1^\infty(u)[\text{stat}] = -2^{l+1} M[l!/(2l)!]\mu\, u^{-1} \qquad (53)$$

which leads to

$$\Psi_1^\infty(u)[\text{stat}] = (-2)^{l+1} \frac{(l!)^2}{(2l)!} M\mu\, u^{-l-1}. \qquad (54)$$

This result is again identical with the one derived by GPP (Eqs. (12) and (18) in [3]) for static initial data.

VII. HIGHER TERMS OF THE EXPANSION

We now turn to evaluate the higher terms ($N \geq 2$) of the iterative expansion, each defined by Eqs. (14—16). These terms may be formally constructed in an inductive way by means of the 'flat space' Green's function derived in sec. V:

$$\Psi_N(u,v) =$$
$$-\int_{u_0}^u du' \int_u^v dv' G(u,v;u',v')\delta V(u',v')\Psi_{N-1}(u',v'). \quad (55)$$

Manifestly, this solution satisfies the initial and boundary conditions specified in Eqs. (15) and (16), respectively.

In analogy to the treatment of Ψ_1 [Eqs. (39),(40)], we can write

$$\Psi_N(u,v) = \frac{\partial^l}{\partial u^l}\left[g_N(u,v)\right], \qquad (56)$$

in which

$$g_N(u,v) = -\int_{u_0}^u du' \int_u^v dv'\, g_G(u;u',v')\delta V(u',v')\Psi_{N-1}(u',v')$$

$$\qquad (57)$$

(with $\Psi_{N-1}(v' < 0) \equiv 0$ for any $N \geq 1$).

With the explicit form of g_G and δV, and following the same integration-by-parts procedure as in deriving Eq. (47), the last equation can be put into the form

$$g_N(u,v) = \sum_{k=0}^l \int_{u_0}^u du' \int_u^v dv' \frac{(v'-u)^l(u-u')^{l-k}}{(v'-u')^{2l-k+3}}$$
$$\times \left[a_k^l + b_k^l \ln(\tilde{v}' - \tilde{u}')\right] g_{N-1}(u',v')\,\theta[r_*' - r_*(R)], \quad (58)$$

in which a_k^l and b_k^l are constant coefficients.

We observe that the analysis of Ψ_N for $N \geq 2$ involves two additional technical difficulties, which were not present in the calculation of Ψ_1. First, the source wave is no longer compact with respect to retarded time, but rather it extends infinitely to the future. This means that Ψ_N^∞ will be influenced by sources located at any value of r, not only at null infinity. (However, as we discuss later, the small-r sources seem not to affect the asymptotic late time behavior at null infinity). The second technical difficulty is the fact that for $N \geq 2$, the source functions $g_{N-1}(u',v')$ are also v'-dependent. Thus, in order to calculate Ψ_N^∞ for $N \geq 2$, one must first be provided with the form of g_{N-1} at *any* value of v, not merely at null infinity.

To proceed, we shall first refer specifically to the case $N = 2$, in order to demonstrate that *the dominant contribution to Ψ_N^∞ at late time comes from Ψ_{N-1} sources at null infinity*; Sources at small r-values do not affect the behavior of the wave at null infinity to the leading order in M/u. Under this assumption, we shall than calculate this dominant contribution, to obtain the late time decay pattern of each of the functions Ψ_N at null infinity. Numerical support shall be presented.

calculation of Ψ_2

To analyze Eq. (58) for $N = 2$, we separate the domain of integration into four regions, labeled I—IV, as indicated in figure 6. In what follows we show that the dominant contribution to $\Psi_2^\infty(u \gg M)$ comes only from sources in region I, which is a distant (large r value) region of spacetime. We shall thus call this zone 'the main' region of integration. An upper bound will be set on the contribution from each of the other regions, II—IV, to verify their relative negligibility.
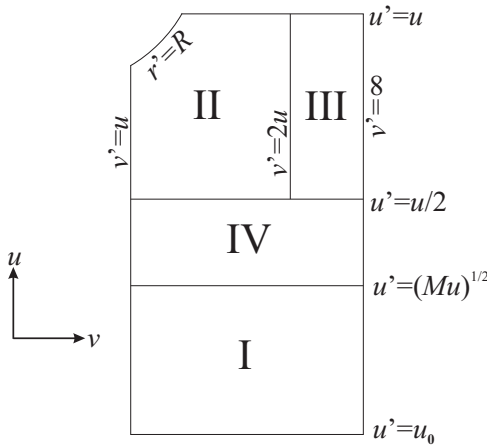
FIG. 6. Range of integration for $\Psi_2^\infty(u)$. The contribution from each of the regions I—IV, indicated in this sketch, is evaluated in the text.

First, consider **region II**, which is covered by $u \le v' \le 2u$ and $u/2 \le u' \le u$, with the sphere $r' < R$ excluded. In this region of integration, the following upper bound is applicable to the source function $g_1(u', v')$:

$$|g_1(u', v')| < CM \frac{(v' - u')^{l+1}}{(u')^{l+3}} \ln(\tilde{v}' - \tilde{u}_0), \qquad (59)$$

where C is a positive constant. To prove this, set $N = 1$ in Eq. (58), with the replacements $(u, v) \to (u', v')$ and $(u', v') \to (u'', v'')$, and with the u'' integration cut off at $u'' = 0$ [since $g_0(u'' > 0) = 0$]. Then use the fact that for any $0 \le k \le l$,

$$\frac{(v'' - u')^l (u' - u'')^{l-k}}{(v'' - u'')^{2l-k+3}} \le \frac{(v'' - u')^l}{(u' - u'')^{l+3}} \le \frac{(v'' - u')^l}{(u')^{l+3}}, \qquad (60)$$

where the first inequality arises since $v'' \ge u'$, and the second is due to the fact that $u'' \le 0$. The inequality (59) then follows from $(\tilde{v}'' - \tilde{u}'') \le (\tilde{v}' - \tilde{u}_0)$ and after integrating over v''.

Now, to set an upper bound to the contribution from region II to Ψ_2^∞, consider Eq. (58) for $N = 2$, with the double integral replaced by $\int_{u/2}^u du' \int_u^{2u} dv'$ (see figure 6). Using the inequality (59), together with $(u - u') \le (v' - u')$ and $(v' - u) \le (v' - u')$, one finds that

$$|g_{2II}^\infty(u)| \le$$
$$C_1 M^2 \ln^2 \tilde{u} \int_{u/2}^u du' \int_{\max\{u, u'+2R\}}^{2u} dv' \frac{(v' - u')^{l-2}}{(u')^{l+3}} \le$$
$$C_2 M^2 u^{-3} \ln^2 \tilde{u}, \qquad (61)$$

in which $g_{2II}^\infty(u)$ stands for the contribution from region II to g_2^∞, and where C_1 and C_2 are positive constants.

Next, consider **region III**. The contribution $g_{2III}^\infty(u)$ arises from the integration $\int_{u/2}^u du' \int_{2u}^\infty dv'$. The upper bound we have set above on $g_1(u', v')$, Eq. (59), in no

longer efficient at large r values. Here, we use a second upper bound, which may be is easily derived from Eq. (58):

$$|g_1(u', v')| \le C_3 M (u')^{-2} \ln \tilde{v}', \qquad (62)$$

where C_3 is yet another positive constant. Provided with this upper bound, we observe that

$$|g_{2III}^\infty(u)| \le C_4 M^2 \int_{u/2}^u du' \int_{2u}^\infty dv' \frac{\ln^2 \tilde{v}'}{(u')^2 (v' - u')^3} \le$$
$$\le C_5 M^2 u^{-3} \ln^2 \tilde{u}, \quad (63)$$

with C_4 and C_5 being positive constants.

We proceed by evaluating the contribution $g_{2IV}^\infty(u)$ from **region IV**. Using again the inequality (62), we find that

$$|g_{2IV}^\infty(u)| \le C_6 M^2 \int_{\sqrt{uM}}^{u/2} du' \int_u^\infty dv' \frac{\ln^2 \tilde{v}'}{(u')^2 (v' - u')^3} \le$$
$$\le C_7 M^2 u^{-5/2} \ln^2 \tilde{u}, \quad (64)$$

where C_6 and C_7 are still another positive constants.

Finally, we turn to calculate the contribution to $\Psi_2^\infty(u)$ due to sources in **region I**, that is at $u_0 \le u' \le \sqrt{Mu}$. To that end we write the source function g_1 in the form

$$g_1(u', v') = g_1^\infty(u') + \Delta_1(u', v'), \qquad (65)$$

in which $\Delta_1(u', v')$ vanishes at null-infinity. In appendix C we show that the contribution to g_2^∞ due to this v'-dependent part of g_1 is bounded from above at large retarded time by

$$|g_{2\Delta}^\infty(u)| \le C_8 M^2 u^{-3} \ln^2 \tilde{u} \qquad (66)$$

where C_8 is a positive constant.

It remains to calculate the contribution to g_{2I}^∞ due to $g_1^\infty(u)$. With the source wave being a function of retarded time only, the analysis is then exactly the same as the above analysis for Ψ_1^∞. In analogy to Eq. (50), we thus immediately obtain, to the leading order in M/u and in u_0/u,

$$g_{2I}^\infty = -2MI_1 u^{-2}, \qquad (67)$$

in which we have defined

$$\int_{u_0}^{\sqrt{Mu}} g_1^\infty(u') du' \simeq \int_{-\infty}^\infty g_1^\infty(u') du' \equiv I_1. \qquad (68)$$

The first equality, accurate to the leading order in M/u, stems from the fact that the integrand is convergent at $u \to \infty$ (since $g_1 \sim u^{-2}$ at large u).

Collecting the above results [Eqs. (61), (63), (64), (66) and (67)] we conclude that

- The dominant contribution to $g_2^\infty(u \gg M)$ [hence also to $\Psi_2^\infty(u \gg M)$] is only due to sources located in the 'main' region, namely at $r \gg M$; Sources at small r values have a negligible effect on g_2^∞ at late time.

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- To calculate the dominant contribution to $g_2^\infty(u \gg M)$, one is allowed to replace $g_1(u', v')$ in Eq. (58) by $g_1^\infty(u)$. This does not affect g_2^∞ to the leading order in M/u.

Therefore, Eq. (67) approximates the 'overall' late time behavior of g_2 at null infinity. For the scalar wave itself we shall thus have, by Eq. (56),

$$\Psi_2^\infty(u \gg m) = 2(-1)^{l+1}(l+1)! M I_1 \, u^{-l-2}, \qquad (69)$$

which is analogous to Eq. (52).

To summarize, we have found that Ψ_2^∞ poses the same late time behavior as Ψ_1^∞, namely a u^{-l-2} inverse power-law decay (for the compact initial data setup). To find out the relative amplitudes of these two terms at late time, one must calculate the ratio of their coefficients, I_1/I_0. This will be done in section VIII, after we first analyze the behavior of the general term Ψ_N of the iterative expansion.

The N'th term of the iterative expansion

We would now like to understand the late time behavior of each of the higher order terms ($N \geq 3$) of the iterative expansion. In principle, this may be done in an inductive way, using Eq. (58). We shall proceed by *assuming* that, as was demonstrated above for Ψ_1 and Ψ_2, the dominant contribution to Ψ_N^∞ at late time (for any $N \geq 1$) is only due to sources at null-infinity. Phrased differently, we adopt the assumption that *late time radiation at null-infinity originates predominantly from scattering off spacetime curvature at null-infinity*. It means, in particular, that it is only the asymptotically-far region of spacetime whose structure affects the asymptotic late-time radiation at null infinity.

Accordingly, in order to obtain the late time form of Ψ_N at null-infinity, we will use Eq. (58) with the source function $g_{N-1}(u', v')$ replaced by $g_{N-1}^\infty(u')$, and with the upper limit of the u'-integration set to $u' = \sqrt{uM}$. This will presumably provide the correct form of $\Psi_N^\infty(u \gg M)$ to the leading order in M/u, as was explicitly indicated in the case $N = 2$. The treatment of the $N \geq 3$ cases is then fully analogous to that of Ψ_2, and the generalization of Eq. (67) is straightforward:

$$g_N^\infty(u \gg M) = -2M I_{N-1} \, u^{-2}, \qquad (70)$$

in which we define

$$I_{N-1} \equiv \int_{-\infty}^{\infty} g_{N-1}^\infty(u') du', \qquad (71)$$

and which is accurate to the leading order in M/u and in u_0/u. By Eq. (56) this finally leads to

$$\Psi_N^\infty(u \gg m) = 2(-1)^{l+1}(l+1)! M I_{N-1} \, u^{-l-2}, \quad (72)$$

which generalizes Eqs. (52) and (69).

We conclude that each of the terms $\Psi_{N \geq 1}$ in the iterative series has a similar late time behavior at null infinity, that is a u^{-l-2} inverse-power law decay. Numerical experiments firmly support this observation, as demonstrated in figure 7. Consequently, one finds that, in general, to obtain the correct overall amplitude of the scalar wave at null infinity (even merely to the leading order in M/u), one must sum all contributions from the various terms $\Psi_{N \geq 1}$. If this sum turns out to converge—then the overall power-law should be u^{-l-2}.
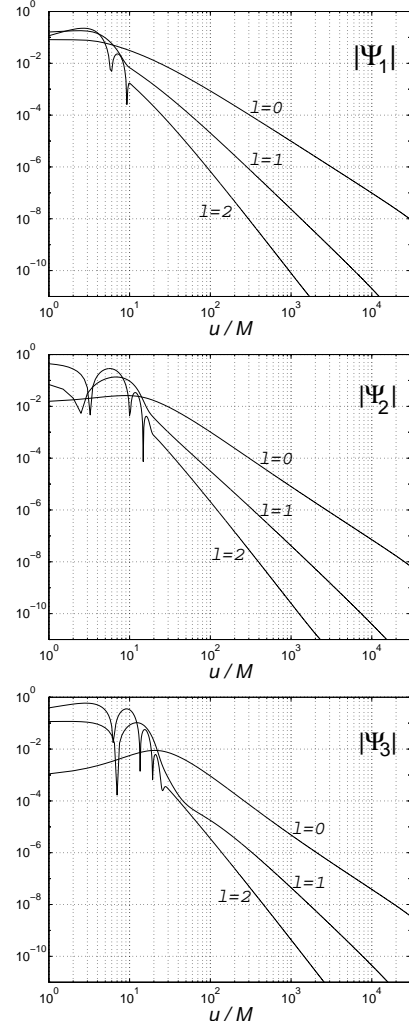
FIG. 7. Late time tails of the iterative expansion terms. Presented on a log-log scale are numerical results obtained for Ψ_1, Ψ_2 and Ψ_3 at $v = 80000M$ (approximating null infinity), for the $l = 0, 1, 2$ modes. Compact initial data for the propagation has been specified between $u = -5M$ and $u = -10M$, and the radius of the shell has been set to $R = 3M$. The results demonstrate the u^{-l-2} late time decay rate predicted by the analytic calculation.

A considerable simplification is achieved when the case $|u_0| \gg M$ is considered. This corresponds to initial setup in which the pulse is specified where spacetime is approximately flat (far outside the highly curved region). In this case, analytic arguments as well as numerical ex-

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periments, both to be presented below, suggest that (i) the iterative series converges at null infinity at late time, and that (ii) the dominating term of the expansion there is Ψ_1, which well approximates the "complete" wave Ψ. (The contribution from the rest of the terms is smaller by order M/u_0).

VIII. CONVERGENCE OF THE ITERATIVE EXPANSION

To discuss the convergence properties of the iterative expansion, we shall concentrate on the case $|u_0| \gg M$. In what follows we argue that the ratio of any two successive terms of the iterative expansion, when evaluated at null infinity at $u \gg M$, is given to the leading order in M/u_0 by

$$\frac{\Psi_{N+1}^\infty}{\Psi_N^\infty} \propto M|u_0|^{-1} \ln|\tilde{u}_0| \qquad (73)$$

(for $N \geq 1$) with a proportion factor depending on N. Below we present numerical results in a firm support of this relation. But first, we show that the validity of the ratio (73) is also suggested using analytic considerations.

Throughout the following discussion we shall refer to the limit $R/u_0 \to 0$ for simplicity. To that end we need to assume that the limit $\Psi_N^\infty(R/u_0 \to 0)$ exist, and that the functions Ψ_N^∞ are each independent of R to the leading order in M/u and in M/u_0 (this is supported by numerical examination).

To evaluate the ratio $\Psi_{N+1}^\infty/\Psi_N^\infty$ we shall have to figure out how the various functions Ψ_N scale with respect to the parameter $|u_0|$. To that end we first show that each of the functions $g_N(u,v)$ can be written in the form

$$g_N(u,v) = u_0^{l-1-N}(\ln|\tilde{u}_0|)^N f_N(\bar{u}, \bar{v}), \qquad (74)$$

where f_N are certain functions of only the dimensionless variables $\bar{u} \equiv u/u_0$ and $\bar{v} \equiv v/u_0$. This form, which is valid globally (for any values of u and v) to the leading order in M/u_0, may be verified using mathematical induction: By Eq. (22) we learn that Eq. (74) holds for g_0.[***] Following the inductive procedure, we now assume that (74) is valid for a certain value of N. We then obtain for g_{N+1} [using Eq. (58) with $x \equiv u'/u_0$ and $y \equiv v'/u_0$],

$$g_{N+1}(u,v) =$$

[***]To explore the scaling of g_0 with respect to u_0, it is convenient to eliminate any scale parameters which may be characteristic of the initial data function $\Gamma(u)$, by referring specifically to the case $\Gamma(u) = \delta(u - u_0)$. Provided that any possible scale characteristic of the initial date is much smaller then $|u_0|$, this cannot limit the generality of our discussion.

$$u_0^{l-2-N}(\ln|\tilde{u}_0|)^N \sum_{k=0}^{l} \int_1^{\bar{u}} dx \int_{\max(0,\bar{u})}^{\bar{v}} dy \frac{(y-\bar{u})^l(\bar{u}-x)^{l-k}}{(y-x)^{2l-k+3}}$$
$$\times \{a_k + b_k \ln[\tilde{u}_0(y-x)]\} f_N(x,y). \qquad (75)$$

For $|u_0| \gg M$ this may be written (neglecting terms which are smaller by $\sim \ln|\tilde{u}_0|$) as

$$g_{N+1}(u,v) = u_0^{l-2-N}(\ln|\tilde{u}_0|)^{N+1} f_{N+1}(\bar{u}, \bar{v}), \qquad (76)$$

where we have defined

$$f_{N+1}(\bar{u}, \bar{v}) \equiv \sum_{k=0}^{l} b_k \int_1^{\bar{u}} dx \int_{\max(0,\bar{u})}^{\bar{v}} dy \frac{(\bar{u}-y)^l(\bar{u}-x)^{l-k}}{(y-x)^{2l-k+3}}. \qquad (77)$$

Thus, by induction, Eq. (74) holds.

In virtue of Eqs. (74) and (70) we then find that (for $N \geq 1$)

$$I_{N-1} \propto u_0^{l+1-N}(\ln|\tilde{u}_0|)^N, \qquad (78)$$

which by Eq. (72) describes the scaling of the functions $\Psi_N^\infty(u \gg M)$ with respect to u_0. The ratio (73) then follows immediately. [We comment that for the $l = 0$ mode, no logarithmic factor is expected to occur neither in Eq. (73) nor in Eq. (78)].

Eq. (73) suggests that the *late time behavior of the scalar wave at null infinity is dominated by* Ψ_1^∞, provided that $|u_0| \gg M$. More accurately, it indicates that to the leading order in M/u, in u_0/u and in M/u_0 we have

$$\Psi^\infty = \Psi_1^\infty. \qquad (79)$$

The significance of this result stems from the fact that we have a simple analytic expression for Ψ_1^∞ at late time.

Numerical investigation, regarding the behavior of the functions Ψ_N^∞ as related to the value of u_0, provides a firm (though qualitative) support to both Eqs. (73) and (79). Examples of these numerical results are presented in figures 8, 9, and 10.

Note that Eq. (73) by itself does not tell us whether the sum of the iterative expansion is convergent, as we do not know the form of the (N-dependent) proportion factor appearing in this expression. However, the numerical experiments strongly suggest that the expansion *does* indeed converge, provided that $|u_0|/M$ is large enough. Also, the rate of convergence seems to increase as we take $|u_0|/M$ to be larger. These features are also apparent in figures 8 through 10.
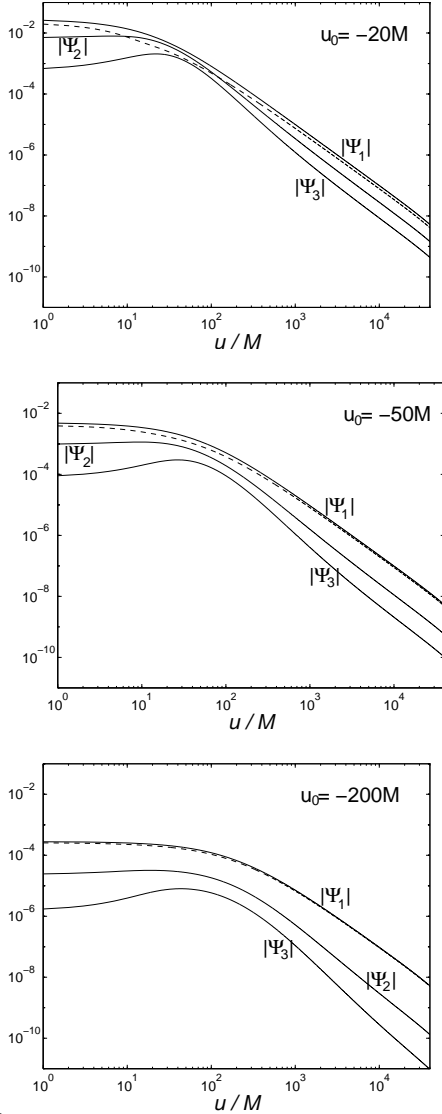
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FIG. 8. Relative amplitudes of the functions Ψ_N at null infinity, for various values of the parameter u_0. Shown are numerical results describing the $l = 0$ mode of Ψ_1, Ψ_2 and Ψ_3 at $v = 10^5 M$ (approximating null infinity), for the cases $u_0 = -20M, -50M, -200M$. Also shown for reference (in dashed line) is the 'complete' wave Ψ, obtained by a direct numerical solution of the wave equation (7). The results support the observation that the ratio $|\Psi_N/\Psi_{N+1}|$ decreases as $|u_0|/M$ is chosen larger, and that for $|u_0|/M$ large enough, Ψ_1^∞ becomes the dominant component of the 'complete' scalar wave Ψ, as suggested by analytic considerations. Note the independence of the amplitude of Ψ_1 in the value of u_0 at late time, which agrees with the scaling rule (78) for the monopole mode (where no logarithmic factor is involved).
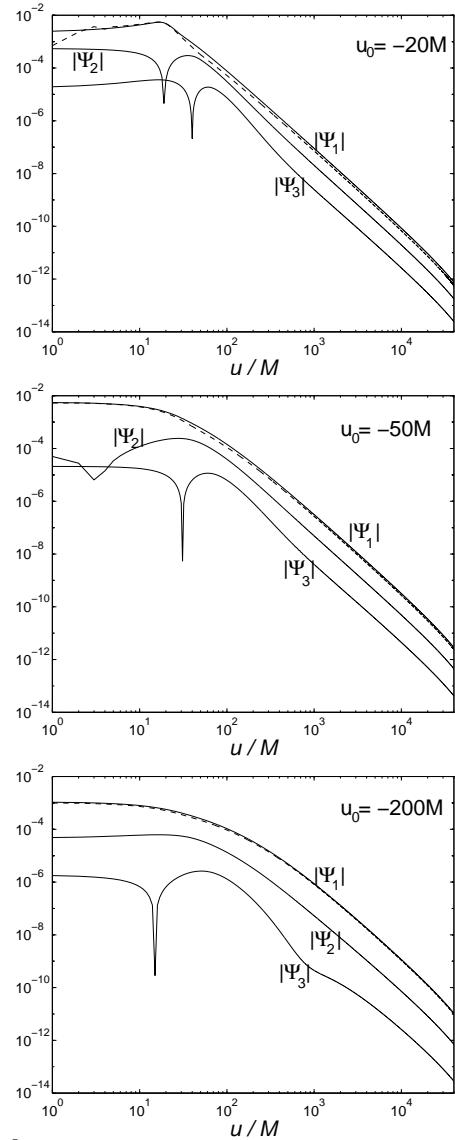

FIG. 9. The same as in figure 8, this time for the $l = 1$ mode of the scalar field. The steep decreases in the amplitude of Ψ_3 occur where this function changes its sign (see the comment in the text). In the case $u_0 = -200M$, a "remnant" of such a sign-changing at $u \simeq 1000M$ "postpones" the development of the power-law tail to later times.
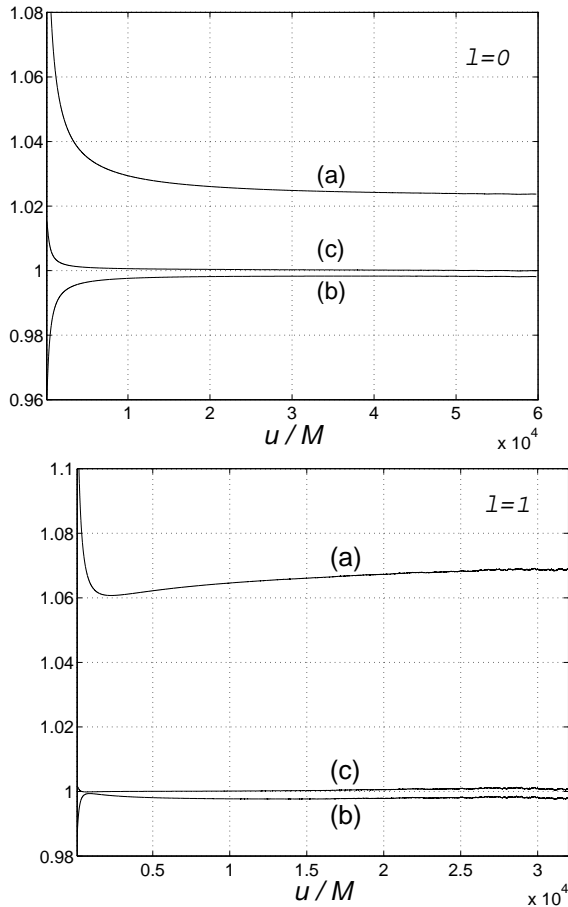
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FIG. 10. Indications for convergence of the iterative scheme. Presented (on a linear scale) are the ratios (a) Ψ_1/Ψ, (b) $\frac{\Psi_1+\Psi_2}{\Psi}$, and (c) $\frac{\Psi_1+\Psi_2+\Psi_3}{\Psi}$ for $l=0$ and for $l=1$. Other parameters are $R=5M$, $u_0=-200M$ and $v=10^5M$ (approximating null infinity). The results demonstrate the convergence of the iterative expansion for large $|u_0|/M$ values at large retarded time u.

The amplitude of the various functions Ψ_N may change its sign during the early stage of evolution (as apparent in figures 7 and 9). Our numerical experiments suggest that this kind of sign-changing occur at larger and larger values of u as N and u_0 increase, "delaying" the formation of power law tails to later and later times. We could not rule out the (somewhat bothering) possibility that for any value of u and u_0, there will exist some N_0 such that for any $N \geq N_0$ sign-changing would occur at retarded time greater than u. In that case, our analytic considerations [especially the ones leading to Eq. (73)] will not apply for all N. Nevertheless, numerical examination shows that the convergence itself is not disturbed by such sign-changing.

Finally, a word of caution is in place: In our analytic calculations we were taking into account only the contributions to the functions Ψ_N due to scattering off the '$r_*^{-3}\ln r_*$' potential, which is the asymptotic form of δV at large distance. We have argued previously that contributions due to the other terms in the M/r_* expansion of δV do not affect the behavior at null infinity at large

u. This, indeed, is supported numerically, as demonstrated in figure 11. However, such terms (each giving an additive contribution to each of the functions Ψ_N) *do* affect the exact amplitude of the leading-order tail: For example, these terms give an additional contribution to the amplitude of Ψ_2^∞, which (in terms of the M/u_0 expansion) is of the same order as the amplitude of Ψ_3^∞. Therefore, to examine whether the sum of the iterative series converges to the actual 'complete' wave Ψ^∞ (at $u \gg M$), one must take into account the 'complete' potential δV. (This, indeed, was done in obtaining the results presented in figures 8 through 10).
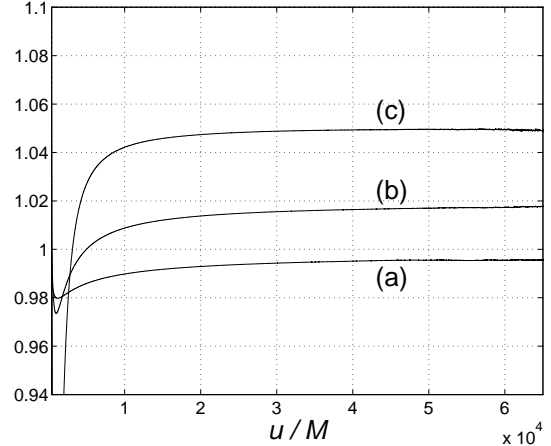

FIG. 11. Contribution from higher order terms in δV to Ψ_N^∞ is negligible at $u \gg M$. This is demonstrated here for the $l=1$ mode in the case $R=3M$ and $u_0=-200M$. Plotted are the ratios (a) Ψ_1/Ψ_1^*, (b) Ψ_2/Ψ_2^*, and (c) Ψ_3/Ψ_3^*, where the functions Ψ_N are those defined with the 'complete' potential δV, and the functions Ψ_N^* are defined with δV taken to the leading order in M/r_*.

IX. SUMMARY AND DISCUSSION

The problem of scalar wave evolution in a curved spherically-symmetric spacetime can be formalized mathematically as a two-dimensional initial-value problem, with a non-trivial effective potential. In this paper we incorporated the simple shell model in order to demonstrate the applicability, and explore the features, of a new calculation scheme to handle this problem. This new scheme is based on a special perturbative decomposition of the wave ("*the iterative expansion*") which, if fact, converts the original homogeneous non-trivial initial-value problem into an infinite hierarchy of inhomogeneous initial-value problems, each having a simple Minkowski-like effective potential. The resulting initial-value problems can then be treated analytically in an inductive manner, using the Green's function in Minkowski spacetime.

Unfortunately, we could not give a full analytic treatment concerning the convergence properties of our scheme. However, some analytic considerations, firmly supported by numerical analysis, indicated that the role

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of the "small parameter" of the iterative expansion is played by M/u_0 (although this parameter is not self-manifested in the wave equations). It was strongly suggested that the iterative expansion *does* converge at null infinity, provided that the support of the initial data for the wave evolution is confined to large distances (small $M/|u_0|$). Moreover, our analysis indicated that under the above condition, the late time behavior of the 'complete' wave is well approximated at null infinity by Ψ_1, namely the first element of the expansion manifested at late time. The last result is of special significance, since the late time form of Ψ_1 at null infinity can be derived easily, as shown in section VI. Hence, for $|u_0| \gg m$ our iterative expansion proves to be a simple effective tool for analyzing the late time behavior of the scalar field at null infinity.

Here, it is important to distinguish between the issue of *efficiency* of the scheme to that of its *usefulness*. The iterative scheme can still prove useful even when $M/|u_0|$ is too large for the expansion to converge efficiently (that is, for Ψ^∞ to be well approximated by merely Ψ_1): In all cases when the scheme converges, it always predicts (by Eq. (72)) that the overall late time tail of Ψ^∞ has the form u^{-l-2} (for the compact initial set-up). The convergence may be 'efficient' (when $|u_0|/M$ is large enough), allowing one to obtain a simple analytic approximation for Ψ^∞ by calculating Ψ_1; or else it may be 'inefficient', in which case our scheme still provides a simple formal way to calculate Ψ^∞, by successive applications of the formula (55) (e. g. using numerical methods).

We think the following is a reasonable physical explanation to the formal relation discovered between the smallness of the parameter $M/|u_0|$ (describing the distance at which the initial perturbation originates), and the effectiveness of the iterative expansion. As already argued by Price [2], spacetime curvature at small distances acts as an (almost) impenetrable potential barrier for the long waves of the scalar field. These long waves are the ones dominating the late time behavior of the field (the effective frequency goes to zero as t goes to infinity). Since long waves coming from large distance cannot reach the highly-curved region (these waves are effectively scattered off the potential tail at large distances), their evolution takes place in the large-r region of spacetime. Thus, for a perturbation originating at large distance ($|u_0| \gg M$), late time behavior of the field should depend mainly on the structure of spacetime at large distance; details of spacetime curvature at the highly-curved (small r) region should be less relevant for this behavior. This may no longer be true if the initial perturbation is specified at small r, thus allowed to evolve through the highly-curved region. Now, the iterative scheme has the mathematical structure of a perturbative expansion, with convergence properties that depend on how much can δV be considered a "small" amount. This function gets arbitrarily small as r gets arbitrarily large, but at small distances it is finite, and not small. However, the above reasoning implies that the form of δV at small dis-

tance seems to be irrelevant to the wave behavior at late time, provided that the initial perturbation originates at large distance ($|u_0| \gg M$). The above argument supplies a physical explanation to our formal result that the effectiveness of the iterative scheme increases as $|u_0|/M$ is taken larger.

In this paper we have focused on analyzing the wave at *null infinity*. The results we obtained have a clear physical significance with respect to a distant static observer: Along world-lines of constant r we have $\Delta u = \Delta t$, where Δu and Δt are, respectively, the retarded time and the (flat-space) static observe's time elapsed since the 'main pulse' of radiation had reached the observer. Now, for r very large, 'null-infinity' is in fact the region $\Delta t \ll r$ [†††]. Thus, in accordance with our results at null infinity, at times $M \ll \Delta t \ll r$ such an observer observes a $\Psi \propto (\Delta t)^{-l-2}$ tail [or a $\Psi \propto (\Delta t)^{-l-2}$ tail in the case of static initial setup].

There is yet another important outcome from the analysis at null infinity: In the accompanying paper we apply a simple method to determine the late time behavior of the scalar wave outside the Schwarzschild black hole at *any constant radius*, down to the event horizon. This is made possible only if one is provided with the form of the wave at null infinity, which shall be calculated in Schwarzschild using the same iterative scheme developed in the present paper. Thus, even thought the iterative scheme seems, in general, not effective at small distances, it still constitute, in our approach, a crucial step in establishing a complete description of the wave behavior at late time.

Generally speaking, we may point out the following advantages of the new technique:

- Most important, the new scheme is the first one directly extensible to realistic rotating black holes, as shall be described in [16]

- The iterative scheme provides a convenient formal framework for studying the effect of the non-trivial curvature-induced part ("δV") of the effective potential on the evolution of the fields. This effect is isolated in our approach by explicitly splitting the wave into its 'trivial' (flat space) part Ψ_0, and a curvature-induced part $\Psi - \Psi_0$, in which each term Ψ_N arises from N scattering events off the curvature potential δV.

- We already discussed the fact that the iterative series can be formally viewed as an expansion of the scalar field in the parameter M/u_0. This feature provides a convenient formal way to explore and quantify the dependence of the wave behavior on

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[†††]The spacetime region $\Delta t << r$, for r very large, may be properly named (after Leaver [9]) 'the *astrophysical zone*' of the waves.

the initial distance of the (compact) perturbation. (an issue somewhat overlooked in previous works.)

Apparently, the iterative scheme developed in this paper can be directly applied to any asymptotically-flat spherically-symmetric spacetime (possessing an effective potential barrier for the waves). That would basically require only the modification of the curvature potential δV to that characterizing the new spacetime under consideration. The results of our analysis were consistent with the general assumption that the small distance details of spacetime structure do not affect the behavior of the wave at asymptotically late time. (This was demonstrated for Ψ_1 and Ψ_2 by an explicit calculation). Thus it seems reasonable to assume that the characteristics of this behavior would be completely determined by merely the large-r asymptotic form of δV.

In the accompanying paper we use the iterative scheme to analyze the behavior of scalar waves at null infinity in the 'complete' Schwarzschild manifold. In this case the details of the analysis become somewhat complicated by the presence of the highly curved region near the horizon. Yet, we shall find that at late time the wave behaves essentially the same as in the shell model, in consistency with the above indications.

APPENDIX A: UNIQUENESS OF THE BOUNDARY CONDITION PROBLEM

The purpose of this appendix is to show that the initial value problem for the scalar wave, with a boundary condition, as defined in Eqs. (7), (9) and (10), has a unique solution. (The formal issue of uniqueness is not trivial because of the divergence of the potential function at the center of symmetry in the 1+1 representation.) It will also be verified that each of the components Ψ_N of our iterative expansion is uniquely defined.

Uniqueness of Ψ_0

First, it will be shown that the solution for Ψ_0 (the "Minkowski-like" first component of the iterative expansion), stated in Eq. (19), is unique. To that end we examine the most general solution to the wave equation (17), given by

$$\Psi_0 = \sum_{n=0}^{l} A_n^l \frac{g_0^{(n)}(u) + (-1)^n h_0^{(n)}(v)}{(v-u)^{l-n}}, \qquad (A1)$$

in which the coefficients A_n^l are those given in Eq. (20), and where $g(u)$ and $h(v)$ are (yet) arbitrary functions. (In flat spacetime, these two functions describe the outgoing and ingoing components of the wave, respectively). It will now be shown that although the functions $g(u)$ and $h(v)$ are not uniquely determined by the initial and boundary conditions, the wave Ψ_0 *is* unique.

Consider first the evolution of the wave between retarded times $u = u_0$ and $u = 0$ (see figure 2 for reference). In this region the wave equation possesses no irregular points, and thus existence and uniqueness of the solution Ψ_0 is guaranteed relying on standard theorems (see, for example, [19]). Therefore, the solution stated in Eq. (19a) is unique at $u \leq 0$.

We thus focus on the region $u \geq 0$, where we have to show that with the conditions $\Psi_0(u = 0) = 0$ and $\Psi_0(r = 0)$ imposed on the general solution $(A1)$, the trivial solution $\Psi_0(u \geq 0) \equiv 0$ is unique. (Due to the linearity of the wave equation, to verify uniqueness it is enough to consider the case in which the wave vanishes on the initial ray $u = 0^{\ddagger\ddagger\ddagger}$. Thus the use of the condition $\Psi_0(u = 0) = 0$ in this context is regardless of the fact that the wave Ψ_0 actually vanishes along the outgoing ray $u = 0$.)

Imposing the condition $\Psi_0(u = 0) = 0$ on the general solution $(A1)$ leads to a linear inhomogeneous ordinary equation for the function $h(v)$, analogous to Eq. (21). By analogy to Eq. (22), the general solution to this equation reads

$$h(v) = \frac{1}{(l-1)!} \sum_{n=0}^{l} A_n^l g_0^{(n)}(0) \, v^{l+1} \int_v^\infty \frac{(v-v')^{l-1}}{(v')^{2l-n+1}} dv'$$
$$+ \sum_{n=l+1}^{2l} \frac{1}{n!} c_n v^n, \qquad (A2)$$

where c_n are l arbitrary parameters. Upon calculating the integral, we find that $h(v)$ is just a polynomial of order $2l$,

$$h(v) = \sum_{n=0}^{2l} \frac{1}{n!} c_n v^n, \qquad (A3)$$

with its first $l + 1$ coefficients given by

$$c_n = -g_0^{(n)}(0) \qquad \text{for } 0 \leq n \leq l. \qquad (A4)$$

Next, we require that Ψ_0 would vanish at $r_* = 0$ (that is for $u = v$). By Eq. (A1) this requirement makes the

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functional equality $g(u) = -h(u)$ compulsory. Note that this result accommodates with Eqs. (A3,A4), allowing both conditions $\Psi_0(u = 0) = 0$ and $\Psi_0(r_* = 0) = 0$ to be satisfied simultaneously without forcing the functions $g(u)$ and $h(v)$ to vanish identically.

We conclude that Ψ_0 has the form

$$\Psi_0 = \sum_{n=0}^{l} A_n^l \frac{\mathcal{P}_l^{(n)}(u) - (-1)^n \mathcal{P}_l^{(n)}(v)}{(v-u)^{l-n}}, \qquad (A5)$$

where $\mathcal{P}_l$ are polynomials of order l [the $(l+1)$th through $(2l)$th powers of the polynomials vanish independently in the summation, in view of Eq. (23)]. Notice that formally, Ψ_0 vanishes along the ray $u = 0$ for any polynomial $\mathcal{P}_l$ whatsoever, as the polynomial coefficients c_n are arbitrary in our construction.

The last remark is important, since it directly implies that Ψ_0 vanishes *anywhere* at $u \geq 0$. For, given a specific value of retarded time $\bar{u} > 0$, we may transform to the new coordinates $x \equiv u - \bar{u}$ and $y \equiv v - \bar{u}$, in terms of which we have

$$\Psi_0 = \sum_{n=0}^{l} A_n^l \frac{\mathcal{P}_l^{(n)}(x + \bar{u}) - (-1)^n \mathcal{P}_l^{(n)}(y + \bar{u})}{(y-x)^{l-n}}$$
$$= \sum_{n=0}^{l} A_n^l \frac{\bar{\mathcal{P}}_l^{(n)}(x) - (-1)^n \bar{\mathcal{P}}_l^{(n)}(y)}{(y-x)^{l-n}}, \qquad (A6)$$

where $\bar{\mathcal{P}}_l$ is still another polynomial of order l. The last expression vanishes at $x = 0$ (for any v) regardless of the form of the polynomial, and thus we find that Ψ_0 vanishes also along the ray $u = \bar{u}$. Since the value of $\bar{u}$ is arbitrary, we are led to the conclusion that Ψ_0 must vanish identically at $u \geq 0$. Therefore, the solution (19) for Ψ_0 is unique.

Uniqueness of the functions Ψ_N

To show that the functions Ψ_N, for $N \geq 1$, are unique, it is enough to verify that the Green's function G is unique. Then, all function Ψ_N are unique by construction.

To discuss the uniqueness of the Green's function, we refer to a source point at (u', v'), as sketched in figure 3. In region I (see the figure) existence and uniqueness are guaranteed by standard theorems [19]. In region II the Green's function admits the general form (A1), as do Ψ_0 (both functions obey the same homogeneous field equation in this region). In the expression corresponding to the Green's function we shall denote the two functions of u and v by $g_G(u)$ and $h_G(v)$ respectively.

The value of the Green's function along the outgoing ray $u = v'$, dictated by the evolution in region I, is unity. Hence, if the boundary condition $G(r_* = 0) = 0$ is to be satisfied, then there has to be a discontinuity in the Green's function at retarded time $u = v = v'$ (the left vertex of the dark-colored rectangle in figure 3).

It can be shown that no solution exist for the Green's function, which is continuous along $u = v'$ (with the single point $u = v = v'$ excluded) and, at the same time, satisfies the boundary condition. For, if we assume that $G(u \to (v')^+) = 1$, then we shall have the function h_G admitting the form

$$h_G(v) = \sum_{n=0}^{2l} \frac{1}{n!} c_n (v - v')^n \qquad (A7)$$

[analogous to Eq. (A3)], where this time the coefficients c_n shall read

$$c_n = \begin{cases} -g_G^{(n)}(v') & , 0 \leq n \leq l - 1 \\ -g_G^{(n)}(v') + 1 & , n = l \end{cases}. \qquad (A8)$$

This cannot be accommodated with the requirement that $h_G(u) = -g_G(u)$, necessary for the boundary condition to hold. Thus the Green's function cannot be continuous along $u = v'$.

Now, the Green's function can still obey the field equation at $u = v'$, provided that the discontinuity along this ray is constant in magnitude for all $v > v'$. This can be verified by writing $G = G \times \theta(u - v') + G \times \theta(v' - u)$, and substituting into the equation for the Green's function. One thereby finds that for this equation to be satisfied at $u = v'$, we must have $G_{,v}(u \to (v')^+) = 0$, and thus $G(u \to (v')^+) = k$, where k is constant. By analogy to Eq. (A8) we find that in this case $h_G^{(l)}(v') = -g_G^{(l)}(v') + k$, which implies that the boundary condition is violated, unless $k = 0$. Hence, the Green's function must "jump" to zero right after retarded time $u = v'$.

Then, by the discussion regarding Ψ_0, we conclude that the only solution to the Green's function in region II, subject to the boundary condition at the origin and continuous everywhere except on the ray $u = v'$, is the trivial solution $G(u > v') \equiv 0$.

Uniqueness of Ψ

Now, consider the initial value (and boundary condition) problem for the "complete" scalar wave Ψ in the shell model. To verify that a solution is unique, again we analyze the situation of zero initial data (on $u = u_0$ and $v = 0$), showing that in this case the null solution ($\Psi \equiv 0$) is unique.

At $u \leq 0$, existence and uniqueness of Ψ is assured, relying on standard theorems. Thus the trivial solution $\Psi(u \leq 0) \equiv 0$ is unique. Particularly, we find that Ψ vanishes along the ray $u = 0$.

Now, consider the region $u \geq 0$, $v \leq 2R$, represented in the diagram of figure 12 by the triangle ABC. In this portion of spacetime (confined to the flat interior of the shell) Ψ obeys the same equation as Ψ_0, and is subject to the boundary condition at $r_* = 0$ and to zero initial condition along the null segment AB (see the figure). By the

above discussion (regarding Ψ_0) we deduce that Ψ must vanish in the portion ABC. In particular, we find that Ψ is zero along the null segment BC. This, in turn, together with the fact that the wave vanishes along the ray BD (see figure), implies (again by standard theorems) that Ψ vanishes anywhere inside the region represented in figure 12 by the rectangle $BDFC$. Thereby we have shown that the wave vanishes anywhere between retarded times $u = 0$ and $u = 2R$.
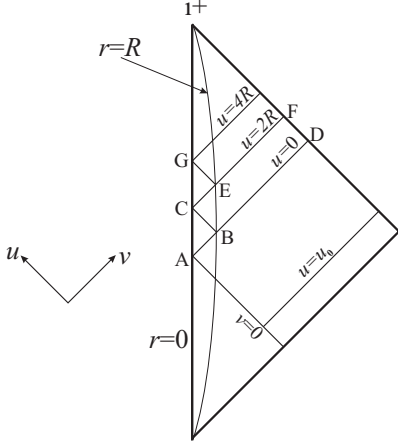

FIG. 12. Uniqueness of the solution for Ψ (see explanation in the text).

We can now proceed, referring next to the triangle CEG (see figure), to show that Ψ vanishes also between retarded times $u = 2R$ and $u = 4R$. The same procedure may be applied an arbitrary number of times, showing that the wave Ψ must vanish anywhere. Since the wave equation is linear, we have thereby shown that the solution for Ψ (with arbitrary initial data and zero boundary condition at the center of symmetry) is unique.

APPENDIX B: DERIVATION OF g_1^∞ FOR $l \geq 1$

In this appendix we give a detailed derivation of $g_1^\infty(u)$ [Eq. (47)] for the case $l \geq 1$. The starting point to our calculation is Eq. (43). It has been shown that the r_*^{-3} potential [namely the term proportional to a in Eq. (41)] gives no contribution to g_1^∞. We therefore set $a = 0$, then integrate Eq. (43) by parts once with respect to v'. Using Eq. (32), this yields

$$g_1^\infty(u) = -\frac{b}{l!} \sum_{n=0}^{l} B_n^l \int_{u_0}^{u} du' g_0^{(n)}(u')(u - u')^{n-2}$$
$$\times \left[\ln(\tilde{v}' - \tilde{u}') + \sum_{m=0}^{l} \frac{l!}{2l - n - m + 2} \right], \quad (B1)$$

with the coefficients B_n^l given in Eq. (45).

To proceed, consider first the logarithmic part of Eq. (B1). We integrate by parts each of the terms in this part

n successive times with respect to u'. As explained previously, compactness of $g_0(u)$ assures that the resulting surface terms would all vanish. One is thus left with

$$-\frac{b}{l!} \sum_{n=0}^{l} B_n^l (-1)^n \int_{u_0}^{0} du' g_0(u') \frac{\partial^n}{\partial u'^n} \left[\frac{\ln(\tilde{v}' - \tilde{u}')}{(u - u')^{2-n}} \right].$$
$$(B2)$$

Summing first the $n = 0$ and $n = 1$ terms yields

$$-\frac{b}{2(2l+1)} \int_{u_0}^{0} \frac{g_0(u')}{(u - u')^2} du', \qquad (B3)$$

in which the vanishing of the logarithmic dependence arises from the equality of B_0^l and B_1^l, which was also responsible for the vanishing of the contribution from scattering off the "r_*^{-3}" potential.

The remaining terms in the sum in Eq. (B2) take the form

$$b \sum_{n=2}^{l} \sum_{j=0}^{n-2} B_n^l \frac{n!(n-2)!(-1)^{n+j}}{j!(n-j-2)!(n-j)} \int_{u_0}^{0} \frac{g_0(u')}{(u - u')^2} du'$$
$$= b \frac{l-1}{4l(l+1)} \int_{u_0}^{0} \frac{g_0(u')}{(u - u')^2} du' \quad (B4)$$

(where the expression given for the sums over j and n is not too difficult to verify, using simple combinatorial manipulations).

The sum of two expressions given in (B3) and (B4) corresponds to the logarithmic part of Eq. (B1). We still have to consider the contribution of the non-logarithmic terms. Integrating these terms by parts n times with respect to u' (with the resulting surface terms again vanishing), we find that only the $n = 0$ and $n = 1$ terms survive, with their sum given by

$$-\frac{b}{2(2l+1)} \sum_{m=0}^{l} \left[\frac{1}{2l - m + 3} - \frac{1}{2l - m + 2} \right]$$
$$\times \int_{u_0}^{0} \frac{g_0(u')}{(u - u')^2} du', \qquad (B5)$$

which is

$$\frac{b}{4(l+1)(2l+1)} \int_{u_0}^{0} \frac{g_0(u')}{(u - u')^2} du'. \qquad (B6)$$

We finally obtain $g_1^\infty(u)$ by collecting the expressions (B3), (B4) and (B6), and setting $b = 8Ml(l+1)$ [by Eq. (42)]. This produces Eq. (47)

APPENDIX C: CONTRIBUTION TO g_2^∞ DUE TO $\Delta_1(u', v')$

The purpose of this appendix is to prove Eq. (66). This equation sets an upper bound on the contribution to g_2^∞

due to the v-dependent part of g_1, namely $\Delta_1(u,v)$ (defined in Eq. (65)).

We first show that the following upper bound is applicable to $\Delta_1(u,v)$ at large r:

$$|\Delta_1(u,v)| \leq CMr_*^{-2}\ln\tilde{r}_*, \qquad (C1)$$

in which C is a positive constant. To verify the validity of this inequality, examine the explicit form of Δ_1, derived directly from Eq. (58),

$$\Delta_1(u,v) = \sum_{k=0}^{l}\int_{u_0}^{0}du'\int_{v}^{\infty}dv'\frac{(v'-u)^l(u-u')^{l-k}}{(v'-u')^{2l-k+3}}$$
$$\times\left[a_k^l + b_k^l\ln(\tilde{v}'-\tilde{u}')\right]g_0(u'). \quad (C2)$$

Now, since $(v'-u)\leq(v'-u')$ and $(u-u')\leq(v'-u')$, we find that there exist positive numbers C', C'', C''' and C, such that

$$|\Delta_1(u,v)| \leq$$
$$C'M\int_{u_0}^{0}du'\int_{v}^{\infty}dv'\frac{\ln(\tilde{v}'-\tilde{u}')}{(v'-u')^3}|g_0^{\infty}(u')| \leq$$
$$C''M\int_{u_0}^{0}du'\frac{\ln(\tilde{v}-\tilde{u}')}{(v-u')^2}|g_0^{\infty}(u')| \leq$$
$$C'''M\frac{\ln(\tilde{v}-\tilde{u})}{(v-u)^2}\int_{u_0}^{0}du'|g_0^{\infty}(u')| \leq$$
$$CMr_*^{-2}\ln\tilde{r}_*, \qquad (C3)$$

which confirms Eq. (C1).

We are now in position to evaluate the contribution of Δ_1 to $g_2^{\infty}(u\gg M)$. By Eqs. (58) and (C1) we deduce that this contribution (in absolute value) is bounded from above by

$$M^2\sum_{k=0}^{l}C\int_{u_0}^{0}du'\int_{u}^{\infty}dv'\frac{(v'-u)^l(u-u')^{l-k}}{(v'-u')^{2l-k+5}}\ln^2(\tilde{v}'-\tilde{u}').$$
$$(C4)$$

Since $(v'-u)\leq(v'-u')$ and $(u-u')\leq(v'-u')$, we find, in turn, that the last expression is bounded from above by

$$C'M^2\int_{u_0}^{0}du'\int_{u}^{\infty}dv'\frac{\ln^2(\tilde{v}'-\tilde{u}')}{(v'-u')^5} \leq$$
$$C''\int_{u_0}^{0}du'\frac{\ln^2(u-u')}{(u-u')^4} \leq CM^2u^{-3}\ln^2\tilde{u}, \quad (C5)$$

for $u\gg M$. [Here, C', C'' and C are some positive numbers, other then in Eq. (C3)].

The inequality (66) is thereby proved.

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